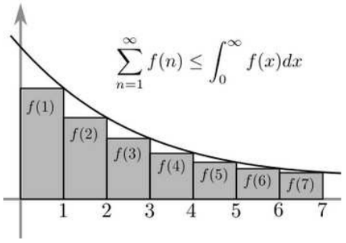
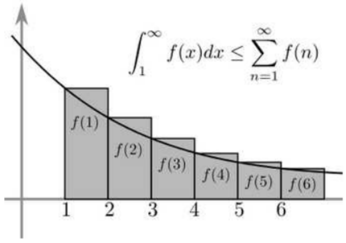
Section 11.3 The Integral Test and Estimation of Sums

The Integral Test: If $f(x)$ is a positive, continuous, decreasing function on $[m, \infty]$, and $a_k = f(k)$, then $\sum_{k=m}^{\infty} a_k$ and $\int_m^{\infty} f(x)\,dx$ either both converge or both diverge.

We can only use the integral test on series whose terms are (eventually) positive and decreasing!





1. Determine whether the following series converge or diverge.

a.) $\sum_{n=1}^{\infty} \frac{n}{n+1}$

Recall: If $\lim_{n\to\infty} a_n \neq 0$, $\sum_{n=1}^{\infty} a_n$ diverges. T.D.

If $\lim_{n\to\infty} a_n = 0$, T.D. fails

$$\lim_{n\to\infty} \frac{n}{n+1} = 1 \neq 0$$

$\boxed{\text{series diverges}}$

b.) $\sum_{n=1}^{\infty} \frac{1}{(7n+8)^3}$

$$\lim_{n\to\infty} \frac{1}{(7n+8)^3} = 0 \quad \text{T.D. Fails}$$

Since $f(x) = \frac{1}{(7x+8)^3}$ is positive and decreasing and easy to integrate, use integral test.

$$\int_1^{\infty} \frac{dx}{(7x+8)^3} = \lim_{t\to\infty} \int_1^t \frac{dx}{(7x+8)^3}$$

$u = 7x+8$

$du = 7\,dx$

$$\frac{1}{7}\int \frac{du}{u^3} = -\frac{1}{14u^2} = -\frac{1}{14(7x+8)^2}$$

$$= \lim_{t\to\infty} \left. \frac{-1}{14(7x+8)^2} \right|_1^t$$

$$= \lim_{t\to\infty} \left(-\cancelto{0}{\frac{1}{14(7t+8)^2}} + \frac{1}{14(15)^2} \right)$$

improper integral converges to $\frac{1}{14(15)^2}$

$\boxed{\text{so } \sum_{n=1}^{\infty} \frac{1}{(7n+8)^3} \text{ converges}}$

c.) $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$ T.D Fails since $\lim_{n\to\infty} \frac{n}{n^2+1} = 0$

$\frac{x}{x^2+1}$ is positive and decreases

easy to integrate

$$\int_1^{\infty} \frac{x}{x^2+1}\,dx = \frac{1}{2}\ln(x^2+1)\Big|_1^{\infty}$$

$$= \frac{1}{2}(\ln\infty - \ln 2)$$

$= \infty$ integral diverges so does series.

d.) $\sum_{n=0}^{\infty} ne^{-n^2}$ $f(x) = xe^{-x^2}$

$$\int_0^{\infty} xe^{-x^2}\,dx = -\frac{1}{2}e^{-x^2}\Big|_0^{\infty} = -\frac{1}{2}(\cancel{e^{-\infty}} \to 0 - 1)$$

$= \frac{1}{2} \rightarrow$ integral converges so does series

e.) $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$ $f(x) = \frac{1}{x\sqrt{\ln x}}$

$u = \ln x$

$du = \frac{dx}{x}$

$$\int \frac{du}{\sqrt{u}} = \frac{u^{\frac{1}{2}}}{\frac{1}{2}}$$

$= 2\sqrt{u}$

$= 2\sqrt{\ln x}$

$$\int_2^{\infty} \frac{dx}{x\sqrt{\ln x}} = 2\sqrt{\ln x}\Big|_2^{\infty}$$

$= 2\sqrt{\ln\infty} - 2\sqrt{\ln 2}$

$= \infty$

integral diverges so does series

We learned in section 7.8 that $\int_1^{\infty} \frac{1}{x^p}\,dx$ will converge if $p > 1$ and will diverge if $p \leq 1$. This gives us the following result:

P-series Test: The **p-series** $\sum\limits_{n=1}^{\infty} \frac{1}{n^p}$, where $p > 0$, is convergent if $p > 1$ and divergent if $p \leq 1$. The special case where $p = 1$ is called the **harmonic series**, which diverges.

A few remarks: Since not all series begin with an index of 1, we can often write $\sum\limits_{n=1}^{\infty} a_n$ simply as $\sum a_n$. We are typically only concerned where the index of the series begins if we are interested in the *sum* of the series. Since convergence of a series is dependent on the *end behavior* of its terms, we can use the notation $\sum a_n$ if we are only interested in the convergence or divergence of the series but not the sum.

2. Determine whether the following series converges or diverges and support your answer.

a.) $\sum\limits_{n=3}^{\infty} \frac{10}{n^{\sqrt{2}}}$ p-series $p=\sqrt{2} > 1$

converges

b.) $\sum\limits_{n=3}^{\infty} \frac{1}{n\sqrt{n}}$ p-series $p = \frac{3}{2} > 1$

converges

c.) $\frac{1}{\sqrt[4]{2}} + \frac{1}{\sqrt[4]{3}} + \frac{1}{\sqrt[4]{4}} + \frac{1}{\sqrt[4]{5}} + \ldots$ p-series $p = \frac{1}{4} < 1$

diverges

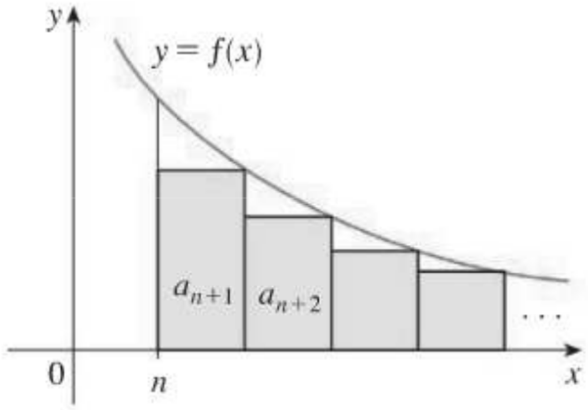
Remainder Estimate for The Integral Test: Suppose $s_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + ... + a_n$ is the n^{th} partial sum of the convergent series $\sum_{n=1}^{\infty} a_n$. Then the **remainder** in using s_n to approximate the sum S is defined to be $R_n = S - s_n = \sum_{i=n+1}^{\infty} a_i = a_{n+1} + a_{n+2} + ...$

$S_n \approx \sum_{n=1}^{\infty} a_n$

Moreover, if $\sum_{n=1}^{\infty} a_n$ was shown to be convergent by the integral test where $a_n = f(n)$, then

$$\boxed{R_n = \sum_{i=n+1}^{\infty} a_i \leq \int_n^{\infty} f(x)\,dx.}$$

$\int_n^{\infty} f(x)\,dx$ is a bound on the remainder (error)



3. For the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$, Find s_6, the sum of the first 6 terms. Using the Remainder Estimate for the Integral Test, estimate the error, R_6, in using the sum of the first 6 terms as an approximation to $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

$S_{\boxed{6}} = a_1 + a_2 + \cdots + a_6$

$= 1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{36}$

Bound on remainder is $\int_6^{\infty} \frac{1}{x^2}\,dx = -\frac{1}{x}\Big|_6^{\infty}$

$= -\frac{1}{\infty} + \frac{1}{6}$

$= \boxed{\frac{1}{6}}$

4. Consider the series $\sum_{n=1}^{\infty} \frac{1}{n^6}$. Using the Remainder Estimate for the Integral Test, find a value of n that will ensure the error in the approximation, s_n, is less than 0.00001. Express your answer as '$n >$_______'.

5. Consider the series $\sum_{n=1}^{\infty} \frac{1}{n^4}$. Using the Remainder Estimate for the Integral Test, find a value of n that will ensure the error in the approximation, s_n, is less than $\frac{1}{95}$. Express your answer as '$n >$_______'.

Once this vaue of n is found, approximate $\sum_{n=1}^{\infty} \frac{1}{n^4}$ to within $\frac{1}{95}$.

Bound on remainder is $\int_n^{\infty} \frac{1}{x^4}\,dx$

$$-\frac{1}{3x^3}\Big|_n^{\infty} = \cancel{-\frac{1}{3(\infty)^3}}^{\,0} + \frac{1}{3n^3}$$

$$= \frac{1}{3n^3}$$

Find n so that $\frac{1}{3n^3} < \frac{1}{95}$

$\boxed{n>4}$ $$\frac{95}{3} < n^3$$ $\boxed{n=4}$