

Always try test for divergence first!

Section 11.4 The Comparison Tests

The Comparison Test: Suppose $\sum_{n=1}^{\infty} a_n$ and $\sum_{i=1}^{\infty} b_i$ are series of positive terms (Note: series need not start at 1.)

given series *comparison series = series of dominating terms*

- If $\sum_{n=1}^{\infty} b_n$ is convergent and $a_n \leq b_n$ for all n , then $\sum_{n=1}^{\infty} a_n$ is also convergent. In other words, "if the larger series converges, so does the smaller series".

- If $\sum_{n=1}^{\infty} b_n$ is divergent and $a_n \geq b_n$ for all n , then $\sum_{n=1}^{\infty} a_n$ is also divergent. In other words, "if the smaller series diverges, so does the larger series".

Note: if the larger series diverges, no conclusion can be made about the smaller series. Likewise, if the smaller series converges no conclusion can be made about the larger series.

- Determine whether the following series converge or diverge. Fully support your answer.

a.) $\sum_{n=1}^{\infty} \frac{5n^4}{n^4 + n^2 + 1}$ T.D. $\lim_{n \rightarrow \infty} \frac{5n^4}{n^4 + n^2 + 1} = 5 \neq 0$
series diverges by T.D.

b.) $\sum_{n=1}^{\infty} \frac{n^4}{n^8 + n^2 + 1}$ T.D. $\lim_{n \rightarrow \infty} \frac{n^4}{n^8 + n^2 + 1} = 0$, T.D. Fails.
positive terms

$$\sum \frac{n^4}{n^8 + n^2 + 1} \leq \sum \frac{n^4}{n^8} = \sum \frac{1}{n^4}$$

converges by p-series $p=4 > 1$
larger converges, so
does smaller by comparison test (CT)

c.) $\sum_{n=3}^{\infty} \frac{n^2 + 1}{\sqrt{n^5 - n}}$

power 2 on top
power $\frac{5}{2}$ on bottom

positive terms

$\left. \rightarrow \text{TD fails} \right\}$

$$\sum \frac{n^2 + 1}{\sqrt{n^5 - n}} \geq \sum \frac{n^2}{\sqrt{n^5}} = \sum \frac{n^2}{n^{\frac{5}{2}}} = \sum n^{-\frac{1}{2}}$$

Smaller diverges
 so does larger by
 CT

d.) $\sum_{n=1}^{\infty} \frac{\sin n + 5}{n^3 \sqrt{n}}$

I suspect convergence, so

a divergent p-series $p = \frac{1}{2} < 1$

$$= \sum_{n=1}^{\infty} \frac{\sin n + 5}{n^{7/2}} \leq \sum_{n=1}^{\infty} \frac{4}{n^{7/2}}, \text{ a convergent p-series}$$

$p = \frac{7}{2} > 1$ larger converges,
so does smaller

I suspect divergence

by CT

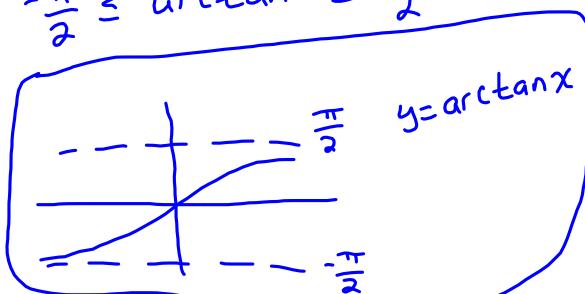
$$e.) \sum_{n=1}^{\infty} \frac{5 + \sin n}{\sqrt{n}} \geq \sum_{n=1}^{\infty} \frac{4}{\sqrt{n}}, \text{ a divergent p-series. } p = \frac{1}{2} < 1$$

smaller diverges
so does larger by CT.

f.) $\sum_{n=1}^{\infty} \frac{5 + \arctan(n)}{n^4}$

$$-\frac{\pi}{2} \leq \arctan n \leq \frac{\pi}{2}$$

$$\frac{5 - \frac{\pi}{2}}{n^4} \leq \frac{5 + \arctan n}{n^4} \leq \frac{5 + \frac{\pi}{2}}{n^4}$$



$\sum \frac{5 + \frac{\pi}{2}}{n^4}$ is a convergent p-series, $p = 4 > 1$
 original series converges
 by CT

$$\text{g.) } \sum_{n=1}^{\infty} \frac{3}{5^n + n}$$

compare with ① $\sum \frac{3}{5^n}$ ← choose this since 5^n dominates n .

$$\text{② } \sum \frac{3}{n} \quad \sum \frac{3}{5^n} = \sum 3\left(\frac{1}{5}\right)^n$$

larger converges
so does smaller
by CT

a convergent
geometric series $r = \frac{1}{5}$
 $|r| < 1$.

$$\text{h.) } \sum_{n=5}^{\infty} \frac{n\sqrt{n}}{8n^2 + 6n} \leq \sum \frac{n\sqrt{n}}{8n^2} = \sum \frac{1}{8\sqrt{n}}$$

larger series diverges
CT fails

a divergent p-series
so CT fails

The Limit Comparison Test: If the Comparison Test is inconclusive, then we may apply the Limit Comparison Test, which determines whether two series behave similarly for very large values of n .

Suppose $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are series of positive terms.

If $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = c > 0$ and finite, then either both series converge or both diverge.

Note: If $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = 0$ or $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \infty$, then the test fails and therefore we need to apply another test.

If $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = c$, $c > 0$ & finite

→ $a_n \sim c b_n$ for large n .

① if $\sum b_n$ converges

then $\sum c b_n$ also converges

$\sum a_n$ also converges

② if $\sum b_n$ diverges, then

$\sum c b_n$ also diverges

$\sum a_n$ also diverges

2. Determine whether the following series converge or diverge.

a.) $\sum_{n=5}^{\infty} \frac{n\sqrt{n}}{8n^2 + 6n}$ LCT failed (From previous problem)

$$a_n = \frac{n\sqrt{n}}{8n^2 + 6n} \quad b_n = \frac{1}{8\sqrt{n}}$$

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \left(\frac{\frac{n\sqrt{n}}{8n^2 + 6n}}{\frac{1}{8\sqrt{n}}} \right) = \lim_{n \rightarrow \infty} \frac{n\sqrt{n}}{8n^2 + 6n} \cdot \frac{8\sqrt{n}}{1} = \lim_{n \rightarrow \infty} \frac{8n^2}{8n^2 + 6n} = 1 > 0$$

both will diverge since $\sum \frac{1}{8\sqrt{n}}$ diverges

b.) $\sum_{n=1}^{\infty} \frac{n^4 - n^3 + 1}{\sqrt{n^{10} - n^6 + 3}}$

LCT: $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \left(\frac{\frac{n^4 - n^3 + 1}{\sqrt{n^{10} - n^6 + 3}}}{\frac{1}{n^4}} \right) = \lim_{n \rightarrow \infty} \left(\frac{n^4 - n^3 + 1}{\sqrt{n^{10} - n^6 + 3}} \cdot n^4 \right) = \lim_{n \rightarrow \infty} \left(\frac{n^8 - n^7 + n^4}{\sqrt{n^{10} - n^6 + 3}} \right) = 1 > 0$

both diverge since $\sum \frac{1}{n}$ diverges

c.) $\sum_{n=1}^{\infty} \frac{n+5}{(2n+1)^3 + 2n}$ Try CT: $\sum \frac{1}{8n^2}$ no inequality is appropriate here, use LCT

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \left(\frac{\frac{n+5}{(2n+1)^3 + 2n}}{\frac{1}{8n^2}} \right) = \lim_{n \rightarrow \infty} \frac{8n^2(n+5)}{(2n+1)^3 + 2n} = 1 > 0$$

so both converge since $\sum \frac{1}{8n^2}$ converges p-series p=2.

d.) $\sum_{n=1}^{\infty} \sin\left(\frac{1}{n^2}\right)$ TD $\lim_{n \rightarrow \infty} \sin\left(\frac{1}{n^2}\right) = \sin(0) = 0 \rightarrow$ Fails

$\sin x \approx x$ near $x=0$

$$\sin\left(\frac{1}{n^2}\right) \approx \frac{1}{n^2} \text{ for large } n$$

LCT with $b_n = \frac{1}{n^2}$

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{\sin\left(\frac{1}{n^2}\right)}{\frac{1}{n^2}} \stackrel{0/0}{=} \lim_{n \rightarrow \infty} \frac{\cos\left(\frac{1}{n^2}\right)}{-\frac{2}{n^3}}$$

$$\lim_{n \rightarrow \infty} \cos\left(\frac{1}{n^2}\right) = 1 > 0$$

both converge since $\sum \frac{1}{n^2}$ converges p-series p=2