Section 11.5 Alternating Series

The convergence tests we have discussed so far apply only to series of positive terms. In this section and the next we learn how to deal with series that are not necessarily positive. Of particular importance are *alternating series*, whose terms alternate signs.

Definition: An alternating series is a series whose terms alternate signs. For example, $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$ is an alternating series. We would like to know under what conditions does an alternating series converge?

The Alternating Series Test: The alternating series $\sum_{n=1}^{\infty} (-1)^n a_n$, where $a_n > 0$, converges if it satisfies both conditions given below:

- $a_{n+1} \leq a_n$ for all n (ie the sequence $\{a_n\}$ is decreasing).
- $\lim_{n\to\infty} a_n = 0$

Illustration as to why this is true.

Consider $\sum_{n=1}^{\infty} (-1)^{n-1} b_n$, where $b_n > 0$ and a_n decreases to zero.

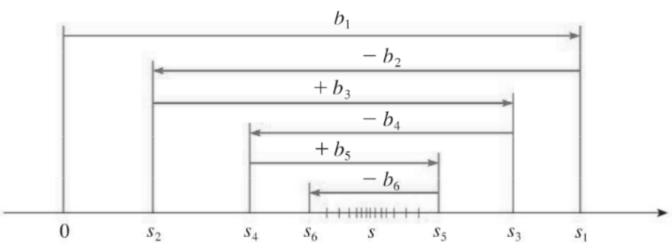
$s_1 = b_1$

$s_2 = b_1 - b_2$

$s_3 = b_1 - b_2 + b_3$

$s_4 = b_1 - b_2 + b_3 - b_4$

$s_5 = b_1 - b_2 + b_3 - b_4 + b_5$



1. Determine whether the following series are convergent.

a.) $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n+3}}$

TD $\lim_{n\to\infty} \frac{(-1)^n}{\sqrt{n+3}} = 0 \rightarrow$ Fails!

Since this is an alternating series do AST

Show $a_n = \frac{1}{\sqrt{n+3}}$ is decreasing ✓ to 0.

$\lim_{n\to\infty} \frac{1}{\sqrt{n+3}} = 0$ ✓

$\boxed{\text{Series converges by AST}}$

b.) $\frac{1}{\ln 2} - \frac{1}{\ln 3} + \frac{1}{\ln 4} - \frac{1}{\ln 5} + ..$

$a_n = \frac{1}{\ln n}$ is decreasing ✓ converges by AST

$\lim_{n\to\infty} \frac{1}{\ln n} = 0$ ✓

c.) $\sum_{n=1}^{\infty} \frac{(-1)^n n^2}{1+n^2}$

$a_n = \frac{n^2}{1+n^2}$ $\lim_{n\to\infty} \frac{n^2}{1+n^2} \neq 0 \longrightarrow$ T.D.

$\lim_{n\to\infty} \frac{(-1)^n n^2}{1+n^2}$ dne $\neq 0$

$\boxed{\text{diverges by TD}}$

d.) $\sum_{n=1}^{\infty} (-1)^{n-1} 2^{-n}$ $= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2^n}$ also a geometric series! $r = \frac{-1}{2}$ $|r|<1$ converge

AST $a_n = \frac{1}{2^n}$ is decreasing

$\lim_{n\to\infty} \frac{1}{2^n} = 0$

will converge by AST

section 11.5 (continued)

Recall : If $\sum_{n=1}^{\infty} (-1)^n a_n$, where $a_n > 0$, then

$\sum_{n=1}^{\infty} (-1)^n a_n$ converges if

- a_n is a decreasing sequence
- $\lim_{n\to\infty} a_n = 0$

e.) $\sum_{n=1}^{\infty} (-1)^n 2^{3/n}$ T.D. $\lim_{n\to\infty} (-1)^n 2^{\frac{3}{n}} \neq 0$ $2^0 = 1$

$\boxed{\text{diverge by TD}}$

f.) $\sum_{n=2}^{\infty} \frac{(-1)^n \ln n}{n}$ T.D. $\lim_{n\to\infty} \frac{(-1)^n \ln n}{n}$

$\lim_{n\to\infty} \frac{\ln n}{n}$ $\frac{\infty}{\infty}$ $\overset{L}{=} \lim_{n\to\infty} \frac{\frac{1}{n}}{1}$ $= 0$

TD Fails!

Use AST : $c_n = \frac{\ln n}{n}$ $\frac{d}{dn}\left(\frac{\ln n}{n}\right) = \frac{\frac{1}{n}(n) - \ln n(1)}{n^2}$ $= \frac{1-\ln n}{n^2} < 0$ for large values of n.

- $\frac{\ln n}{n}$ is decreasing
- $\lim_{n\to\infty} \frac{\ln n}{n} = 0$ (we showed that above)

$\therefore$ $\sum_{n=2}^{\infty} \frac{(-1)^n \ln n}{n}$ converges by AST

Recall: Remainder estimate for integral test:

error or remainder $= R_n < \int_n^\infty f(x)\,dx \quad \left(\text{For } \sum_{n=1}^{\infty} a_n\right)$

Remainder Estimate and The Alternating Series Estimation Theorem

If $\sum_{n=1}^{\infty} (-1)^n a_n$, $\boxed{a_n > 0,}$ is a convergent alternating series, and a partial sum $s_n = \sum_{i=1}^{n} (-1)^i a_i$ is used to approximate the sum of the series with remainder R_n, then

$$|\text{error}| = |R_n| \leq a_{n+1}$$

2. Consider $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3}$

a.) Prove the series is convergent.

AST: $c_n = \frac{1}{n^3}$ — clearly decreasing; $\lim_{n\to\infty} \frac{1}{n^3} = 0$

converges by AST

b.) Use s_6 to approximate the sum of the series and use the Alternating Series Estimation Theorem to estimate the error in using the 6th partial sum to approximate the sum of the series.

$S_{\boxed{6}} =$ sum of first 6 terms

$= -1 + \frac{1}{2^3} - \frac{1}{3^3} + \frac{1}{4^3} - \frac{1}{5^3} + \frac{1}{6^3} - \frac{1}{7^3}$ "next term"

Formula: $|R_n| \leq c_{n+1} \longrightarrow |R_6| \leq c_7 = \frac{1}{7^3}$ absolute value of next term

c.) Determine the minimum number of terms we need to add in order to find the sum with error less than $\frac{1}{150}$.

Find n, n: integer! Find n

$|\text{error}| \leq |R_n| < \boxed{c_{n+1} < \frac{1}{150}}$ $\rightarrow c_{n+1} = \frac{1}{(n+1)^3}$

$\frac{1}{(n+1)^3} < \frac{1}{150} \longrightarrow 150 < (n+1)^3$

$n=4$: $150 < 5^3$ no

$n=5$: $150 < 6^3$ $\boxed{\text{yes}}$

$\boxed{n=5}$

d.) estimate $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3}$ with error $< \frac{1}{150}$

$$\boxed{\text{use } S_5 = -1 + \frac{1}{2^3} - \frac{1}{3^3} + \frac{1}{4^3} - \frac{1}{5^3}}$$

3. approximate $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ to within an accuracy of 10^{-2}.

$$|R_n| < C_{n+1} = \frac{1}{\sqrt{n+1}} < \frac{1}{100}$$

$$100 < \sqrt{n+1}$$

$$10,000 < n+1$$

$$9999 < n$$

Since 9999 is an integer,

use S_{9999}