

Section 11.6 Absolute Convergence and the Ratio Test

Given any series, $\sum_{n=1}^{\infty} a_n$, we can consider the corresponding series $\sum_{n=1}^{\infty} |a_n| = |a_1| + |a_2| + |a_3| + \dots$

We learned in section 11.5 that $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ and $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ both converge by the alternating series test. What happens if we consider:

$$\text{a.) } \sum_{n=1}^{\infty} \left| \frac{(-1)^n}{n} \right| = |-1| + \left| \frac{1}{2} \right| + \left| -\frac{1}{3} \right| + \left| \frac{1}{4} \right| + \left| -\frac{1}{5} \right| + \dots$$

$$= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \dots$$

$$= \sum_{n=1}^{\infty} \frac{1}{n}$$

diverges by p-series.
so we say $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ is conditionally convergent

$$\text{b.) } \sum_{n=1}^{\infty} \left| \frac{(-1)^n}{n^2} \right| = \sum_{n=1}^{\infty} \frac{1}{n^2} \text{ converges by p-series}$$

$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ is absolutely convergent

Definition: A series is **absolutely convergent** if $\sum_{n=1}^{\infty} |a_n|$ converges. If $\sum_{n=1}^{\infty} a_n$ converges, but $\sum_{n=1}^{\infty} |a_n|$ diverges, then the $\sum_{n=1}^{\infty} a_n$ is **conditionally convergent**. Note: If $\sum_{n=1}^{\infty} a_n$ is a series of positive terms, and $\sum_{n=1}^{\infty} a_n$ converges, then by default it is absolutely convergent.

If a series is absolutely convergent, then it is convergent.

1. Determine whether the following series are absolutely convergent, conditionally convergent, or divergent.

a.) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n\sqrt{n}}$

TD Fails

if being asked if a series converges absolutely, do not do AST because AST will only establish convergence.

"Absolute convergence test": $\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{n\sqrt{n}} \right| = \sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}}$
 $\sum_{n=1}^{\infty} \frac{(-1)^n}{n\sqrt{n}}$ converges absolutely
 converges by p-series, $p = \frac{3}{2} > 1$

b.) $\sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n}$

TD Fails!

Look at $\sum_{n=2}^{\infty} \left| \frac{(-1)^n}{n \ln n} \right| = \sum_{n=2}^{\infty} \frac{1}{n \ln n}$

IT: $\int_2^{\infty} \frac{dx}{x \ln x} = \ln|\ln x| \Big|_2^{\infty}$

$u = \ln x$
 $du = \frac{1}{x} dx$

$\int \frac{1}{u} du = \ln|u| = \ln|\ln x|$

$= \ln|\ln \infty| - \ln|\ln 2|$

$= \infty \rightarrow \sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n}$

does not converge absolutely

but it could converge conditionally.

do AST:

$L_n = \frac{1}{n \ln n}$ (decreasing)
 $\lim_{n \rightarrow \infty} \frac{1}{n \ln n} = 0$
 $\sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n}$ converges conditionally

c.) $\sum_{n=1}^{\infty} \frac{\cos n}{n^4}$

TD Fails

cannot do comparison test on $\sum \frac{\cos n}{n^4}$ because it is not positive!

absolute convergence?

$\sum_{n=1}^{\infty} \left| \frac{\cos n}{n^4} \right| \leq \sum_{n=1}^{\infty} \frac{1}{n^4}$

convergent p-series $p=4 > 1$

larger converges, so does smaller by CT

$\therefore \sum_{n=1}^{\infty} \frac{\cos n}{n^4}$ converges absolutely

The Ratio Test:

- If $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent (and therefore convergent).
- If $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = L > 1$ or $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.
- If $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = 1$, then the test fails.

only use ratio test if series contains factorials and/or exponentials (a^n)

2. Determine whether the following series are absolutely convergent, conditionally convergent, or divergent.

a.) $\sum_{n=1}^{\infty} \frac{3^n}{n^3(-2)^n}$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{3^{n+1}}{(n+1)^3(-2)^{n+1}} \cdot \frac{n^3(-2)^n}{3^n} \right|$$

$$= \lim_{n \rightarrow \infty} \left| \frac{3 \cdot \cancel{3^n}}{(n+1)^3(-2)(-\cancel{2})^n} \cdot \frac{\cancel{n^3}(-2)^n}{\cancel{3^n}} \right|$$

$$= \lim_{n \rightarrow \infty} \left| \frac{3}{-2(n+1)^3} \right|$$

$$= \frac{3}{2} > 1 \quad \boxed{\text{series diverges}}$$

b.) $\sum_{n=1}^{\infty} \frac{n^{10}(-100)^n}{n!}$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(n+1)^{10}(-100)^{n+1}}{(n+1)!} \cdot \frac{n!}{n^{10}(-100)^n} \right|$$

$6! = \underbrace{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}_{5!}$

$$= \lim_{n \rightarrow \infty} \left| \frac{(n+1)^{10}(-100)(-\cancel{100})^n}{(n+1)\cancel{n!}} \cdot \frac{\cancel{n!}}{\cancel{n^{10}}(-100)^n} \right|$$

$$= \lim_{n \rightarrow \infty} \left| \frac{(n+1)^{10}(-100)}{(n+1)n^{10}} \right| = 0 < 1$$

converges (absolutely)

c.) $\sum_{n=1}^{\infty} \frac{(2n+1)!}{n!}$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(2n+3)!}{(n+1)!} \cdot \frac{n!}{(2n+1)!} \right|$$

$2(n+1)+1$

$$= \lim_{n \rightarrow \infty} \left| \frac{(2n+3)(2n+2)\cancel{(2n+1)!}}{(n+1)\cancel{n!}} \cdot \frac{\cancel{n!}}{\cancel{(2n+1)!}} \right|$$

$$= \lim_{n \rightarrow \infty} \left| \frac{(2n+3)(2n+2)}{n+1} \right| = \infty > 1$$

diverges!

3. For which of the following series is the ratio test inconclusive?

a.) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ RT $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(-1)^{n+1}}{n+1} \cdot \frac{n}{(-1)^n} \right|$

$$= \lim_{n \rightarrow \infty} \left| \frac{(-1)^n}{n+1} \right| = |-1| = 1 \rightarrow$$

RT Fails

b.) $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$

RT is conclusive

since $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = 0$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(-1)^{n+1} n!}{(n+1)! (-1)^n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(-1)^{n+1}}{(n+1)\cancel{n!}} \cdot \frac{\cancel{n!}}{(-1)^n} \right| = 0$$

c.) $\sum_{n=1}^{\infty} \frac{(-1)^n n!}{(n+1)!}$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n \cancel{n!}}{(n+1)\cancel{n!}}$$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(-1)^{n+1} (n+1)!}{(n+2) (-1)^n} \right|$$

= 1

RT will fail

test fails