

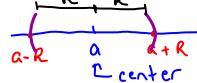
Section 11.8 Power Series

Definition: A Power Series is a series of the form $\sum_{n=1}^{\infty} c_n(x-a)^n$, where x is the variable and the c_n 's are called the coefficients of the series. More generally, $\sum_{n=1}^{\infty} c_n(x-a)^n$ is called a power series centered at $x=a$, or a power series about a . [Further, the interval of convergence is the set of all x for which the power series converges, denoted by I .] We define the radius of convergence to be $R = \frac{1}{2}$ (length of I).

Theorem: For a given power series $\sum_{n=1}^{\infty} c_n(x-a)^n$ there are only three possibilities:

- (i) The series converges only for $x=a$. In this case, $R=0$ and $I=\{a\}$. [Strong c_n]
- (ii) The series converges for all x . In this case, $R=\infty$ and $I=(-\infty, \infty)$. [Weak (small) c_n]

(iii) There is a positive number R such that the series converges if $|x-a| < R$ and diverges if $|x-a| > R$. In this case, R the radius of convergence and the interval of convergence is AT LEAST $I = (-R+a, R+a)$. We must test the endpoints for convergence.



1. For the following power series, find the radius and interval of convergence.

a.) $\sum_{n=0}^{\infty} \frac{3^n x^n}{n^2 + 1}$ center is 0

Apply Ratio Test: $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{3^{n+1} x^{n+1}}{(n+1)^2 + 1} \cdot \frac{n^2}{3^n x^n} \right|$

$$\begin{aligned} &= \lim_{n \rightarrow \infty} \left| \frac{3 \cdot 3^n x^n}{(n+1)^2 + 1} \cdot \frac{n^2}{3^n x^n} \right| \\ &= \lim_{n \rightarrow \infty} \left| \frac{3x(n^2+1)}{(n+1)^2+1} \right| \end{aligned}$$

$$|3x|$$

Since we are looking for interval of convergence,

the ratio test says

we get convergence

if $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$

Step 2: $|3x| < 1$

$$3|x| < 1$$

$$R = \frac{1}{3}$$

$$\begin{aligned} |x| &< \frac{1}{3} \\ -\frac{1}{3} &< x < \frac{1}{3} \\ \text{diverges} &\quad C \quad \text{converges} \quad \text{diverges} \\ -\frac{1}{3} &\quad 0 \quad \frac{1}{3} \end{aligned}$$

Step 3: Test endpoints for convergence

$$\sum_{n=0}^{\infty} \frac{3^n x^n}{n^2 + 1}$$

$x = -\frac{1}{3} \rightarrow \sum_{n=0}^{\infty} 3^n \left(-\frac{1}{3}\right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{n^2 + 1}$

included in interval of convergence

AST: $C_n = \frac{1}{n^2 + 1}$

converges by AST is decreasing

$$\lim_{n \rightarrow \infty} \frac{1}{n^2 + 1} = 0$$

Conclusion

$$R = \frac{1}{3}$$

$$I = \left[-\frac{1}{3}, \frac{1}{3} \right]$$

include in interval

$$\sum_{n=0}^{\infty} \frac{3^n \left(\frac{1}{3}\right)^n}{n^2 + 1} = \sum_{n=0}^{\infty} \frac{1}{n^2 + 1}$$

CT

larger conv by p-series $p=2 > 1$

b.) $\sum_{n=1}^{\infty} \frac{(2x-1)^n}{4n-1}$ center is $\frac{1}{2}$ since $2x-1 = 2(x-\frac{1}{2})$

$$\text{RT } \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(2x-1)(2x-1)}{4(n+1)-1} \cdot \frac{4(n-1)}{(2x-1)^n} \right|$$

$$= \lim_{n \rightarrow \infty} \left| \frac{(2x-1)(4n-1)}{4n+3} \right| \xrightarrow{\frac{4}{4}=1}$$

$$= |2x-1| < 1$$

$$= \left| 2\left(x - \frac{1}{2}\right) \right| < 1$$

$$= \left| x - \frac{1}{2} \right| < \frac{1}{2} \quad R = \frac{1}{2}$$

Test endpoints

$$\sum_{n=1}^{\infty} \frac{(2x-1)^n}{4n-1}$$

$$-\frac{1}{2} < x - \frac{1}{2} < \frac{1}{2}$$

$$0 < x < 1$$

$$x=0: \sum_{n=1}^{\infty} \frac{(-1)^n}{4n-1} \quad \text{AST: } C_n = \frac{1}{4n-1} \quad \text{decreasing} \leftarrow$$

converges

$$x=1: \sum_{n=1}^{\infty} \frac{1}{4n-1} \geq \sum_{n=1}^{\infty} \frac{1}{4n} \quad \begin{array}{l} \text{smaller} \\ \text{diverges} \end{array} \quad \begin{array}{l} \text{larger} \\ \text{also} \\ \text{diverges} \end{array}$$

by p-series $p=1$

CT

include in interval

$R = \frac{1}{2}$

$I = [0, 1)$

$$\text{c.) } \sum_{n=0}^{\infty} \frac{(2n)!(2x-5)^n}{(2n)! (2x-5)^n} \quad \text{RT: } \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(2n+2)!(2x-5)(2x-5)}{(2n)!(2x-5)} \right|$$

(~~2n!~~)! $(2(n+1))!$

$$= \lim_{n \rightarrow \infty} \left| \frac{(2n+2)(2n+1)(2n)! (2x-5)}{(2n)!} \right|$$

$$= \lim_{n \rightarrow \infty} |(2n+2)(2n+1)(2x-5)| = \infty$$

$R = 0$

$I = \left\{ \frac{5}{2} \right\}$

UNLESS $x = \frac{5}{2}$

in which case
the limit is
 $0 < 1$
by RT will converge

only if $x = \frac{5}{2}$

$$\begin{aligned}
 \text{d.) } & \sum_{n=0}^{\infty} \frac{(x+1)^n}{(2n+1)!} \\
 \stackrel{RT}{\lim}_{n \rightarrow \infty} & \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(x+1)(x+1)}{(2n+3)!} \cdot \frac{(2n+1)!}{(x+1)^n} \right| \\
 & = \lim_{n \rightarrow \infty} \left| \frac{x+1}{(2n+3)(2n+2)(2n+1)!} \cdot \frac{(2n+1)!}{1} \right| \\
 & = 0 < 1 \quad \text{for } \underline{\underline{x}} \leq x \\
 & \boxed{I = (-\infty, \infty), R = \infty}
 \end{aligned}$$

$$\begin{aligned}
 \text{e.) } & \sum_{n=2}^{\infty} \frac{(-1)^n (x+1)^n}{2^n \ln n} \\
 \stackrel{RT}{\lim}_{n \rightarrow \infty} & \left| \frac{(-1)(x+1)(x+1)}{2 \cdot 2 \ln(n+1)} \cdot \frac{2 \ln n}{(-1)(x+1)} \right| \\
 & \lim_{n \rightarrow \infty} \left| \frac{(-1)(x+1) \ln n}{2 \ln(n+1)} \right| \xrightarrow{1} \frac{\ln n}{\ln(n+1)} \\
 & \left| \frac{-1}{2} (x+1) \right| = \left| -\frac{1}{2} \right| |x+1| \quad x+1 = x - (-1) \\
 & \quad C = -1
 \end{aligned}$$

Test endpoints

$$\sum_{n=2}^{\infty} \frac{(-1)^n (x+1)^n}{2^n \ln n}$$

$$x=1: \sum_{n=2}^{\infty} \frac{(-1)^n 2^n}{2^n \ln n}$$

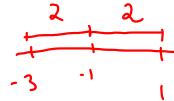
$$\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n} \quad \text{converges by AST} \quad c_n = \frac{1}{\ln n} \quad \text{decreases to 0}$$

$$|x+1| < 2$$

$$\boxed{R=2}$$

$$-2 < x+1 < 2$$

$$-3 < x < 1$$



$$x=-3: \sum_{n=2}^{\infty} \frac{(-1)^n (-2)^n}{2^n \ln n} = \sum_{n=2}^{\infty} \frac{2^n}{2^n \ln n} = \sum_{n=2}^{\infty} \frac{1}{\ln n} > \sum_{n=2}^{\infty} \frac{1}{n}$$

Fact $\ln n < n$ 

$$I = [-3, 1]$$

diverges by CT

2. Suppose it is known that $\sum_{n=0}^{\infty} c_n x^n$ converges when $x = -4$ and diverges when $x = 6$. What can be said about the convergence or divergence of the following series:

a.) $\sum_{n=0}^{\infty} c_n (2)^n$ C

b.) $\sum_{n=0}^{\infty} c_n (8)^n$ D

c.) $\sum_{n=0}^{\infty} c_n (4)^n$ no conclusion

d.) $\sum_{n=0}^{\infty} c_n (-5)^n$ no conclusion

it diverges at $x=8$
because if it converges
at $x=8$, then R is at least
8, meaning it converges
for $-8 < x < 8$. But we were
told it diverged at 6.

3. Suppose it is known that $\sum_{n=0}^{\infty} c_n (x-1)^n$ converges when $x = 3$. On what interval are we certain the series converges?

since 1 is the center, and it converges at $x=3$, R is at least 2.

$(-1, 3]$