Section 11.9 Representations of Functions as Power Series

In this section we learn how to represent certain types of functions as sums of power series by manipulating geometric series or by differentiating or integrating such a series.

We learned the geometric series $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$, provided $|x| < 1$. Since this power series is centered at 0, this means the radius of convergence is $R = 1$, which is equivalent to the interval $(-1, 1)$.

If we test the endpoints for convergence, we will find divergence at both endpoints since both $\sum_{n=0}^{\infty}(1)^n$ and $\sum_{n=0}^{\infty}(-1)^n$ diverge by the Test for Divergence. We therefore have the following result:

$\boxed{\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \text{ with a radius of convergence } R = 1 \text{ and interval of convergence } I = (-1, 1).}$

1. Find a power series representation for the function and determine the radius and interval of convergence.

a.) $f(x) = \frac{1}{1-3x} = \sum_{n=0}^{\infty} (3x)^n$, $\quad |3x| < 1$

$$\boxed{\frac{1}{1-3x} = \sum_{n=0}^{\infty} 3^n x^n}$$

$|x| < \frac{1}{3}$

$$\boxed{\begin{array}{l} R = \frac{1}{3} \\ I = \left(-\frac{1}{3}, \frac{1}{3}\right) \end{array}}$$

b.) $f(x) = \frac{1}{5+4x} = \frac{1}{5\left(1+\frac{4x}{5}\right)} = \frac{1}{5}\left(\frac{1}{1-\left(-\frac{4x}{5}\right)}\right)$

$$= \frac{1}{5}\sum_{n=0}^{\infty}\left(-\frac{4x}{5}\right)^n, \quad \left|-\frac{4x}{5}\right| < 1$$

$$= \frac{1}{5}\sum_{n=0}^{\infty}\frac{(-1)^n 4^n x^n}{5^n} \qquad \frac{4}{5}|x| < 1$$

$$|x| < \frac{5}{4}$$

$$\boxed{\frac{1}{5+4x} = \sum_{n=0}^{\infty}\frac{(-1)^n 4^n x^n}{5^{n+1}}} \qquad \boxed{R = \frac{5}{4}, \quad I = \left(-\frac{5}{4}, \frac{5}{4}\right)}$$

c.) $f(x) = \frac{x}{2-x^2} = \frac{x}{2\left(1-\frac{x^2}{2}\right)} = \frac{x}{2}\left(\frac{1}{1-\frac{x^2}{2}}\right)$

$$= \frac{x}{2}\sum_{n=0}^{\infty}\left(\frac{x^2}{2}\right)^n, \quad \left|\frac{x^2}{2}\right| < 1$$

$$= \frac{x}{2}\sum_{n=0}^{\infty}\frac{x^{2n}}{2^n} \qquad |x^2| < 2$$

$$|x| < \sqrt{2}$$

$$\boxed{\frac{x}{2-x^2} = \sum_{n=0}^{\infty}\frac{x^{2n+1}}{2^{n+1}}} \qquad \boxed{\begin{array}{l} R = \sqrt{2} \\ I = (-\sqrt{2}, \sqrt{2}) \end{array}}$$

Theorem If $f(x) = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \ldots + c_n x^n + \ldots$ has a radius of convergence R.
Then

a.) $f'(x) = c_1 + 2c_2 x + 3c_3 x^2 + \ldots + nc_n x^{n-1} + \ldots$ has a radius of convergence R.

$$f'(x) = \sum_{\boxed{n=1}}^{\infty} c_n n x^{n-1} = \sum_{n=0}^{\infty} c_{n+1}(n+1)x^n$$

$$\sum_{n=1}^{\infty} a_n = \sum_{n=0}^{\infty} a_{n+1}$$

b.) $\int f(x)\,dx = C + c_0 x + \frac{c_1 x^2}{2} + \frac{c_2 x^3}{3} \ldots + \frac{c_n x^{n+1}}{n+1} + \ldots$ has a radius of convergence R.

$$\int f(x)\,dx = C + \sum_{n=0}^{\infty} \frac{c_n}{n+1} x^{n+1}$$

Note: When **differentiating** a power series, the starting index **will** change if the first term is constant.
When **integrating** a power series, the starting index does **not** change under any circumstances.

2. If $f(x) = \sum_{n=0}^{\infty} \underbrace{\frac{(n+1)}{n!}}_{c_n} x^n$, find $f'(x)$ and $\int f(x)\,dx$.

$$f'(x) = \sum_{n=1}^{\infty} \frac{n+1}{n!} n x^{n-1} = \sum_{n=0}^{\infty} \frac{n+2}{(n+1)!}(n+1)x^n$$

$$\int f(x)dx = C + \sum_{n=0}^{\infty} \frac{\cancel{n+1}}{n!} \frac{x^{n+1}}{\cancel{n+1}} = \sum_{n=0}^{\infty} \frac{n+2}{\cancel{(n+1)}n!} \cancel{(n+1)} x^n$$

$$= \boxed{C + \sum_{n=0}^{\infty} \frac{x^{n+1}}{n!}} \qquad \boxed{f'(x) = \sum_{n=0}^{\infty} \frac{n+2}{n!} x^n}$$

3. Evaluate $\int \frac{1}{1+x^6}\,dx$ as a power series. $= \int \frac{1}{1-(-x^6)} dx$

$$= \int \sum_{n=0}^{\infty} (-x^6)^n dx, \quad \boxed{|x|<1 \quad R=1}$$

$$= \int \sum_{n=0}^{\infty} (-1)^n x^{6n} dx$$

$$= \boxed{C + \sum_{n=0}^{\infty} \frac{(-1)^n x^{6n+1}}{6n+1}}$$

4. Find a power series representation for the function and determine the radius of convergence.

a.) $\frac{1}{(9+x)^2}$

if you see $\frac{1}{(\ )^2}$ First integrate it

$$\int \frac{1}{(9+x)^2}dx = \frac{-1}{9+x} = \frac{-1}{9\left(1+\frac{x}{9}\right)} = -\frac{1}{9}\left(\frac{1}{1-\left(-\frac{x}{9}\right)}\right)$$

$$= -\frac{1}{9}\sum_{n=0}^{\infty}\left(-\frac{x}{9}\right)^n, \quad \boxed{R=9}$$

$$= -\frac{1}{9}\sum_{n=0}^{\infty}\frac{(-1)^n x^n}{9^n}$$

$$\frac{d}{dx}\int \frac{1}{(9+x)^2}dx = \frac{d}{dx}\sum_{n=0}^{\infty}\frac{(-1)^{n+1}x^n}{9^{n+1}}$$

$$\frac{1}{(9+x)^2} = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{9^{n+1}}n x^{n-1}$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^{n+2}}{9^{n+2}}(n+1)x^n$$

$$= \boxed{\sum_{n=0}^{\infty}\left(-\frac{1}{9}\right)^{n+2}(n+1)x^n}$$

what if $\frac{x^2}{(1-2x^3)^2}$

b.) $\frac{x^2}{(1-2x)^2} = x^2\left(\frac{1}{(1-2x)^2}\right)$

Step 1: $\int \frac{1}{(1-2x)^2}dx = \frac{1}{2}\left(\frac{1}{1-2x}\right)$

$$= \frac{1}{2}\sum_{n=0}^{\infty}(2x)^n \qquad R=\frac{1}{2}$$

$$= \frac{1}{2}\sum_{n=0}^{\infty}2^n x^n$$

$$\frac{d}{dx}\int \frac{1}{(1-2x)^2}dx = \frac{d}{dx}\sum_{n=0}^{\infty}2^{n-1}x^n$$

$$\frac{1}{(1-2x)^2} = \sum_{n=1}^{\infty}2^{n-1}n x^{n-1}$$

$$= \sum_{n=0}^{\infty}2^n(n+1)x^n$$

Step 2: multiply both sides by x^2

$$\frac{x^2}{(1-2x)^2} = x^2\sum_{n=0}^{\infty}2^n(n+1)x^n$$

$$= \boxed{\sum_{n=0}^{\infty}2^n(n+1)x^{n+2}}$$

c.) $\ln(3+x)$ If you see ln(sum) take derivative

$$\frac{d}{dx}\ln(3+x) = \frac{1}{3+x} \qquad \frac{d}{dx}\ln(\text{sum}) = \frac{(\text{sum})'}{\text{sum}}$$

$$= \frac{1}{3(1+\frac{x}{3})}$$

$$= \frac{1}{3}\sum_{n=0}^{\infty}\left(-\frac{x}{3}\right)^n$$

$$\int \frac{d}{dx}\ln(3+x)\,dx = \int \sum_{n=0}^{\infty}\frac{(-1)^n x^n}{3^{n+1}}\,dx \qquad R=3$$

$$\ln(3+x) = C + \sum_{n=0}^{\infty}\frac{(-1)^n}{3^{n+1}}\frac{x^{n+1}}{n+1}$$

to find C, let $x=0$

$$\ln 3 = C + \sum 0 \rightarrow C = \ln 3$$

$$\ln(3+x) = \ln 3 + \sum_{n=0}^{\infty}\frac{(-1)^n x^{n+1}}{3^{n+1}(n+1)}$$

d.) $x\ln(3+x)$

$$x\ln(3+x) = x\left(\ln 3 + \sum_{n=0}^{\infty}\frac{(-1)^n x^{n+1}}{3^{n+1}(n+1)}\right)$$

$$= \boxed{x\ln 3 + \sum_{n=0}^{\infty}\frac{(-1)^n x^{n+2}}{3^{n+1}(n+1)}}$$

e.) $\int x\ln(3+x)\,dx = \int \left(\quad \right) dx$

$$= C + \frac{x^2}{2}\ln 3 + \sum_{n=0}^{\infty}\frac{(-1)^n}{3^{n+1}(n+1)}\frac{x^{n+3}}{n+3}$$

f.) $\arctan x$

$$\int \frac{d}{dx} \arctan x \, dx = \frac{1}{1+x^2} = \sum_{n=0}^{\infty} \left(-x^2\right)^n = \int \sum_{n=0}^{\infty} (-1)^n x^{2n} dx, \quad R=1$$

$$\arctan x = C + \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$$

choose $x=0$

$$\arctan 0 = C + \sum 0 \rightarrow C=0$$

$$\boxed{\arctan \boxed{x} = \sum_{n=0}^{\infty} \frac{(-1)^n \boxed{x}^{2n+1}}{2n+1}}$$

g.) $\arctan\left(\frac{x}{2}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{x}{2}\right)^{2n+1}}{2n+1} = \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)2^{2n+1}}}$

h.) $\int_0^{1/3} \arctan\left(\frac{x}{2}\right) dx = \int_0^{\frac{1}{3}} \quad dx$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{(2n+1)2^{2n+1}(2n+2)} \Bigg|_0^{\frac{1}{3}}$$

$$= \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{1}{3}\right)^{2n+2}}{(2n+1)2^{2n+1}(2n+2)}} - \sum 0$$