Section 5.5: Integration by Substitution

The Substitution Rule: If $u = g(x)$ is a differentiable function, then

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du$$

Note: Typically, u is chosen so that du is a factor of the integrand.

1. $\int 2t^2(t^3-1)^3\,dt$

$u = t^3 - 1$

$\frac{du}{dt} = 3t^2 \longrightarrow du = 3t^2\,dt$

$\boxed{\frac{du}{3} = t^2\,dt}$

$$2\int \underbrace{(t^3-1)^3}_{u^3}\,\underbrace{t^2\,dt}_{\frac{du}{3}}$$

$$\frac{2}{3}\int u^3\,du = \frac{\cancel{2}}{3}\frac{u^4}{\cancel{4}\,2} + C = \frac{1}{6}u^4 + C$$

$$= \boxed{\frac{1}{6}(t^3-1)^4 + C}$$

2. $\int \frac{\sec^2 x}{\sqrt{\tan x + 9}}\,dx$ du u

$u = \tan x + 9$

$du = \sec^2 x\,dx$

$$\int \frac{du}{\sqrt{u}} = \int u^{-\frac{1}{2}}\,du$$

$$= \frac{u^{\frac{1}{2}}}{\frac{1}{2}} + C = 2\sqrt{u} + C$$

$$= \boxed{2\sqrt{\tan x + 9} + C}$$

3. $\int_0^{13} \frac{1}{\sqrt[3]{(1+2x)^2}}\,dx$

$\cancel{u = (1+2x)^2}$

$$\int_{0=x}^{13=x} \underbrace{(1+2x)^{-\frac{2}{3}}}_{u^{-\frac{2}{3}}} \underbrace{dx}_{\frac{du}{2}}$$

$u = 1+2x$ $\quad x=13,\ u=27$; $\quad x=0,\ u=1$

$du = 2\,dx$

$\frac{du}{2} = dx$

$$\frac{1}{2}\int_1^{27} u^{-\frac{2}{3}}\,du$$

$$\frac{1}{2}\,\frac{u^{\frac{1}{3}}}{\frac{1}{3}}\Big|_1^{27} = \frac{3}{2}\left(27^{\frac{1}{3}} - 1^{\frac{1}{3}}\right)$$

$27^{\frac{1}{3}} = \sqrt[3]{27} = 3$

$$= \frac{3}{2}(3-1) = \boxed{3}$$

4. $\int \frac{x^2}{(1-x)^4}\,dx$

$u = 1-x \rightarrow x = 1-u \rightarrow x^2 = (1-u)^2$

$du = -1\,dx$

$dx = -du$

$$-\int \frac{x^2}{u^4}\,du = -\int \frac{(1-u)^2}{u^4}\,du = -\int \frac{1-2u+u^2}{u^4}\,du$$

$u = 1-x$

$$= -\int\left(\frac{1}{u^4} - \frac{2u}{u^4} + \frac{u^2}{u^4}\right)du$$

$$= -\int\left(u^{-4} - 2u^{-3} + u^{-2}\right)du$$

$$= -\left(\frac{u^{-3}}{-3} - \frac{\cancel{2}u^{-2}}{-\cancel{2}} + \frac{u^{-1}}{-1}\right) + C$$

$$= -\left(\frac{-1}{3(1-x)^3} + \frac{1}{(1-x)^2} - \frac{1}{1-x}\right) + C$$

Recall: $\int e^u\,du = e^u + C$

5. $\int \frac{e^{1/x}}{x^2}\,dx$ $\quad -du$

$u = \frac{1}{x}$

$du = -\frac{1}{x^2}\,dx$

$-du = \frac{1}{x^2}\,dx$

$u = x^{-1}$

$du = -x^{-2}\,dx$

$du = -\frac{1}{x^2}\,dx$

$$-\int e^u\,du$$

$$-e^u + C$$

$$\boxed{-e^{\frac{1}{x}} + C}$$

6. $\int_0^{\pi/4} e^{\sin(2t)} \cos(2t)\, dt$

$u = \sin(2t)$ $\begin{cases} t = \frac{\pi}{4},\ u = \sin\frac{\pi}{2} = 1 \\ t = 0,\ u = \sin 0 = 0 \end{cases}$

$du = \cos(2t) \cdot 2$

$\frac{du}{2} = \cos(2t)\,dt$

$\frac{1}{2}\int_0^1 e^u\, du$

$\frac{1}{2} e^u \Big|_0^1$

$\frac{1}{2}(e^1 - e^0) = \boxed{\frac{1}{2}(e-1)}$

7. $\int \tan(x)\, dx = \int \frac{\sin x}{\cos x}\, dx$

$u = \cos x$

$du = -\sin x\, dx$

$= -\int \frac{du}{u}$

recall: $a \ln b = \ln b^a$

$= -\ln|u| + C$

$= \boxed{-\ln|\cos x| + C}$

$= \ln|\cos x|^{-1} + C$

$= \ln\left|\frac{1}{\cos x}\right| + C = \boxed{\ln|\sec x| + C}$

8. $\int \frac{\arctan x}{1+x^2}\, dx$

$u = \arctan x$

$du = \frac{1}{1+x^2}\, dx$

$\int u\, du = \frac{u^2}{2} + C$

$= \frac{(\arctan x)^2}{2} + C$

9. $\int \frac{x+1}{x^2+1}\,dx = \int \left(\frac{x}{x^2+1} + \frac{1}{x^2+1} \right) dx$

$$= \underbrace{\int \frac{x}{x^2+1}\,dx}_{} + \underbrace{\int \frac{1}{x^2+1}\,dx}_{\arctan x}$$

$u = x^2+1$

$du = 2x\,dx$

$\frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|u|$

$= \frac{1}{2}\ln|x^2+1|$

Final answer: $\frac{1}{2}\ln|x^2+1| + \arctan x + C$

10. $\int_{e^3}^{e^4} \frac{1}{x\ln x}\,dx = \int_{e^3}^{e^4} \frac{1}{\ln x}\,\frac{1}{x}\,dx$

$u = \ln x$ $\begin{cases} x = e^4,\ u = 4 \\ x = e^3,\ u = 3 \end{cases}$

$du = \frac{1}{x}\,dx$

$= \int_3^4 \frac{1}{u}\,du$

$= \ln|u|\Big|_3^4 = \ln 4 - \ln 3 = \boxed{\ln\frac{4}{3}}$

Note: If the choice of u is linear (degree 1), then du is a constant multiple of dx and hence can be divided out of the integral. To illustrate, let's find $\int e^{kx}\,dx$.

$\int e^{kx}\,dx$

$u = kx$

$du = k\,dx$

$\int \cos(kx)\,dx$

$\frac{1}{k}\int e^u\,du = \frac{1}{k}e^u + C$

$\frac{1}{k}\sin(kx) + C$

$= \frac{1}{k}e^{kx} + C$

11. $\int \left(\cos(5x) + e^{-10x} - \frac{3}{4x-1}\right) dx$

$\frac{1}{5}\sin(5x) + \frac{-1}{10}e^{-10x} - \frac{3}{4}\ln|4x-1| + C$

12. $\int (\sin(3\alpha) - \sin(3x))\, dx$

13. $\int \frac{\sin x}{\sqrt{1-\cos^2 x}}\, dx$

14. $\int \frac{x}{1+x^4}\, dx$