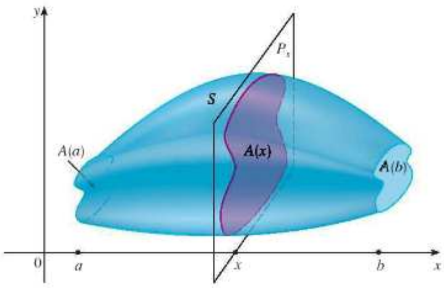
Section 6.2: Volume

In this section, we will learn how to find the volume of a solid by cutting the solid into a set of cross-sectional 'slices'. If we have a formula for the volume of each slice, a Riemann sum will approximate the total volume, and therefore an integral will give the exact volume of the solid.



Definition: let S be a solid that lies between $x = a$ and $x = b$. If the cross-sectional area of S passing through x and perpendicular to the x-axis is $A(x)$, then the volume of S is

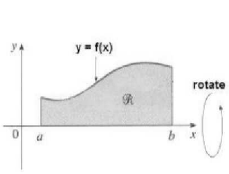
$$V = \int_a^b A(x)\, dx$$

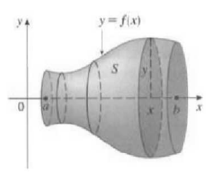
Note: When we use this formula, it is important to remember that $A(x)$ is the *area* of a moving cross-section obtained by slicing the solid through x perpendicular to the x-axis.

Volumes of solids of revolution

Disk Method: Use when the cross-section of the solid is in the shape of a disk.

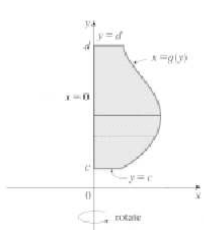
• Rotation around the x axis:

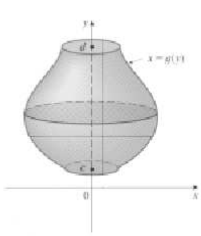




$$V = \int_a^b \pi [f(x)]^2\, dx$$

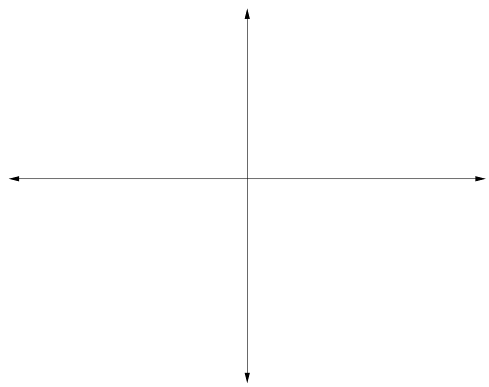
• Rotation around the y axis:



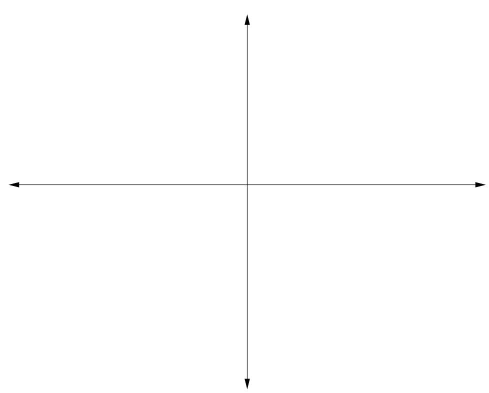


$$V = \int_c^d \pi [g(y)]^2\, dy$$

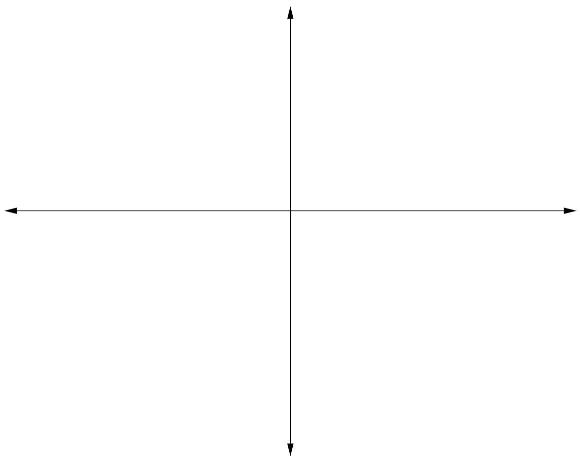
1. Sketch the region R bounded by $y = e^x$, $x = -1$, $x = 2$, $y = 0$. Find the volume of the solid obtained by R about the x axis.



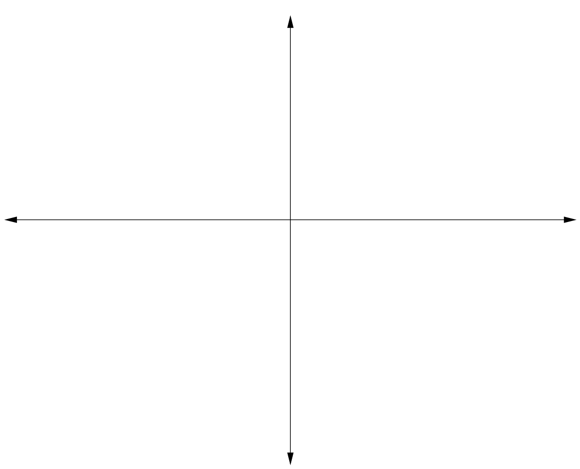
2. Sketch the region R bounded by $y = x^3$, $y = 27$, $x = 0$. Find the volume of the solid obtained by rotating R about the y axis.



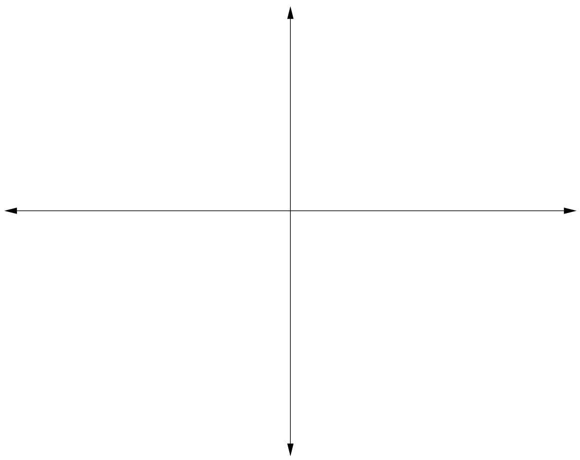
3. Sketch the region R bounded by $y = \ln x$, $x = 0$, $y = 0$, $y = 5$. Find the volume of the solid obtained by rotating R about the y axis.



4. Sketch the region R bounded by $y = \sqrt{x}$, $y = 0$, $x = 2$. Set up but do not evaluate an integral that finds the volume generated by rotating R about the line $x = 2$.

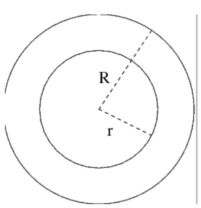


5. Sketch the region R bounded by $y = x^2 + 1$ and $y = 2$. Find the volume generated by rotating R about the line $y = 2$.

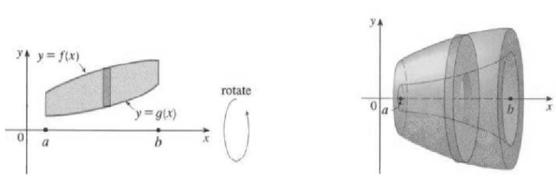


Washer Method: Use when the cross-section of the solid is in the shape of a washer.

Consider the washer shown here:

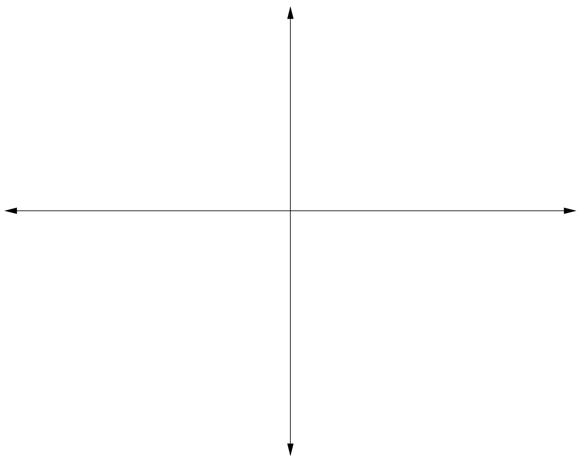


The area of this washer is $A = \pi R^2 - \pi r^2 = \pi(R^2 - r^2)$.

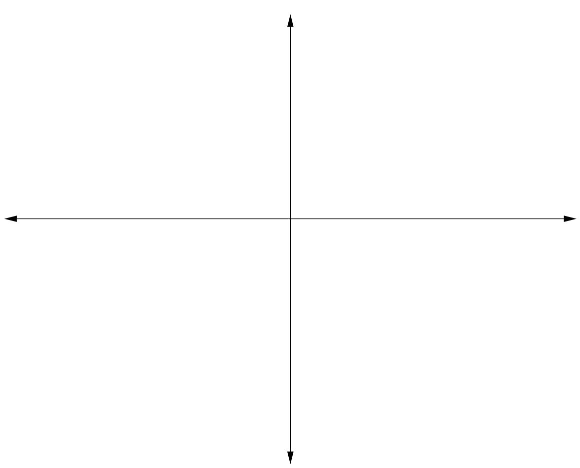


For the image above, $R = f(x)$ and $r = g(x)$, therefore $V = \int_a^b \pi \left([f(x)]^2 - [g(x)]^2 \right) \, dx$.

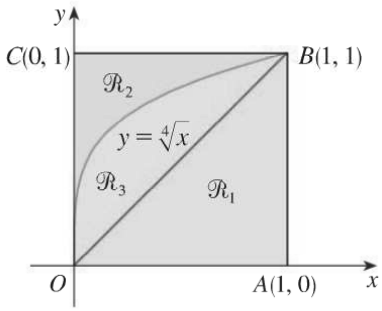
6. Sketch the region R bounded by $y = x^2$, $y = 2x$. Find the volume generated by rotating R about the y axis.



7. Sketch the region R bounded by $y = x$, $y = x^2$. Find the volume generated by rotating R about the line $y = -1$.



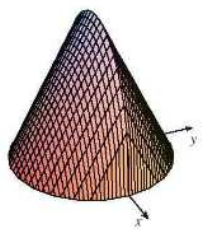
8. Refer to the figure below to set up but do not evaluate an integral that finds the volume generated by rotating the given region about the specified line.

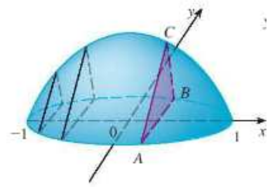


(a) R_2 about BC

(b) R_3 about AB

The Method of Slicing: Here, the solid is not the result of a revolution. Rather, the solid is defined by describing the base (bottom) of the solid, and the shape of a cross-section perpendicular to one of the coordinate axes. For example, in the illustration below, the base of the solid is an elliptical region and cross-sections perpendicular to the x-axis are equilateral triangles. To find the volume, integrate the area of a cross-section, $A(x)$ in this case, for $-1 \leq x \leq 1$, that is $V = \int_{-1}^{1} A(x)\,dx$, where $A(x)$ is the area of the equilateral triangle for $-1 \leq x \leq 1$.





(a) The solid

9. Find the volume of S where the base of S is the region bounded by $y = x^2$ and $y = \sqrt{x}$. The cross sections perpendicular to the x-axis are squares.

10. Find the volume of the solid S whose base is the ellipse $x^2 + 4y^2 = 1$. The cross sections of S perpendicular to the y-axis are squares.

11. Find the volume of the solid S whose base is the triangular region with vertices $(0, 0)$, $(1, 0)$ and $(0, 2)$. The cross sections of S perpendicular to the x-axis are semi-circles.

12. Find the volume of the solid S whose base is the region bounded by the parabola $y = x^2$ and $y = 1$. The cross sections of S perpendicular to the y-axis are equilateral triangles.