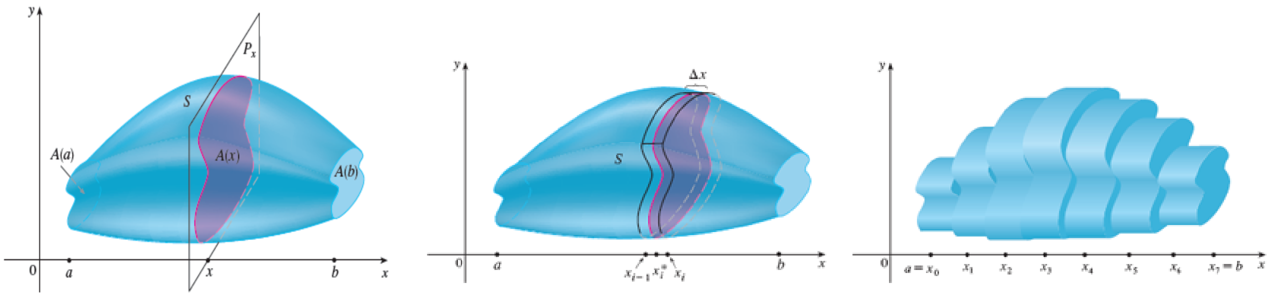
Section 6.2: Volume

In this section, we will learn how to find the volume of a solid by cutting the solid into a set of cross-sectional 'slices'. If we have a formula for the volume of each slice, a Riemann sum will approximate the total volume, and therefore an integral will give the exact volume of the solid.



Definition: let S be a solid that lies between $x = a$ and $x = b$. If the cross-sectional area of S passing through x and perpendicular to the x-axis is $A(x)$, then the volume of S is

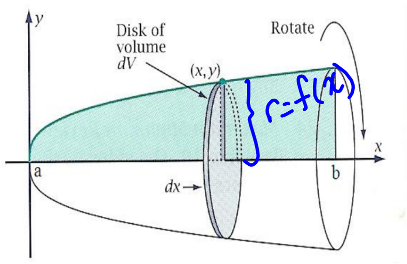
$$V = \int_a^b A(x)\,dx$$

Note: When we use this formula, it is important to remember that $A(x)$ is the *area* of a moving cross-section obtained by slicing the solid through x perpendicular to the x-axis.

Volumes of solids of revolution

Disk Method: Use when the cross-section of the solid is in the shape of a disk.

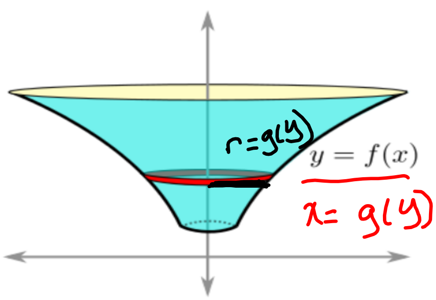
- Rotation around the x axis:



$$A(x) = \pi r^2$$
$$= \pi (f(x))^2$$

$$V = \int_a^b \pi [f(x)]^2\,dx$$

- Rotation around the y axis:

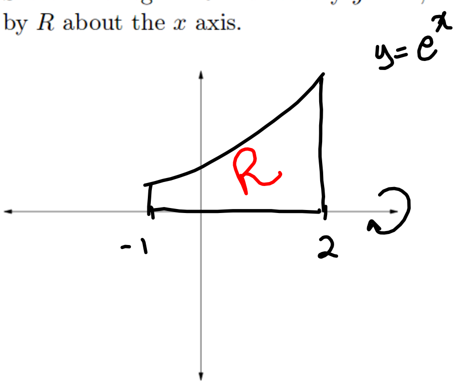


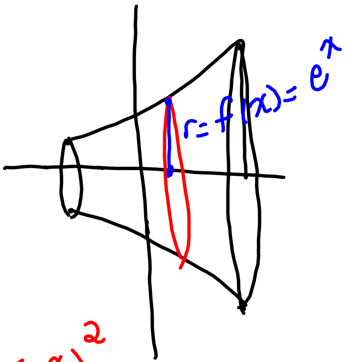
$x = g(y)$

$$A(y) = \pi r^2$$
$$= \pi (g(y))^2$$

$$V = \int_c^d \pi [g(y)]^2\,dy$$

1. Sketch the region R bounded by $y = e^x$, $x = -1$, $x = 2$, $y = 0$. Find the volume of the solid obtained by R about the x axis.





Disk method: $A(x) = \pi\left(e^x\right)^2$

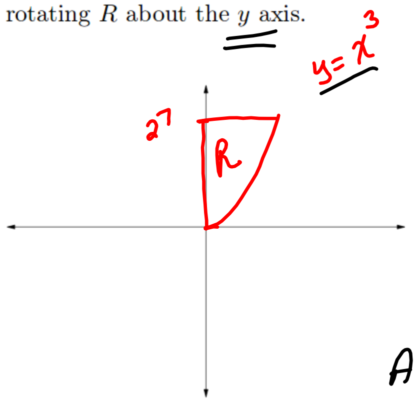
$A(x) = \pi e^{2x}$

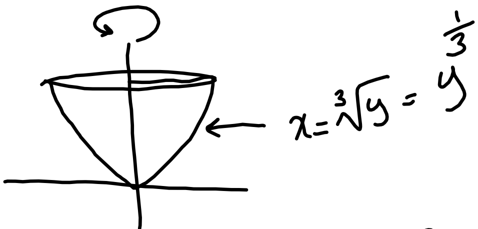
$$V = \int_{-1}^{2} \pi e^{2x}\,dx$$

$$= \pi \cdot \frac{1}{2} e^{2x} \Big|_{-1}^{2}$$

$$\boxed{V = \frac{\pi}{2}\left(e^4 - e^{-2}\right)}$$

2. Sketch the region R bounded by $y = x^3$, $y = 27$, $x = 0$. Find the volume of the solid obtained by rotating R about the y axis.





$$A(y) = \pi r^2 = \pi\left(y^{\frac{1}{3}}\right)^2 = \pi y^{\frac{2}{3}}$$

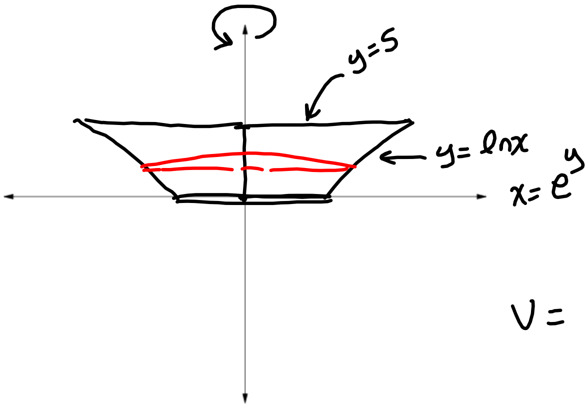
$$V = \int_0^{27} \pi y^{\frac{2}{3}}\,dy = \pi \frac{3}{5} y^{\frac{5}{3}} \Big|_0^{27}$$

$$= \frac{3\pi}{5}\left(27^{\frac{5}{3}} - 0^{\frac{5}{3}}\right)$$

$$= \frac{3\pi}{5}\left(3^5\right)$$

$$= \frac{729\pi}{5}$$

3. Sketch the region R bounded by $y = \ln x$, $x = 0$, $y = 0$, $y = 5$. Find the volume of the solid obtained by rotating R about the y axis.



$$A(y) = \pi (e^y)^2$$

$$A(y) = \pi e^{2y}$$

$$V = \int_0^5 \pi e^{2y}\, dy$$

$$= \frac{\pi}{2} e^{2y} \Big|_0^5$$

$$= \boxed{\frac{\pi}{2}(e^{10} - 1)}$$

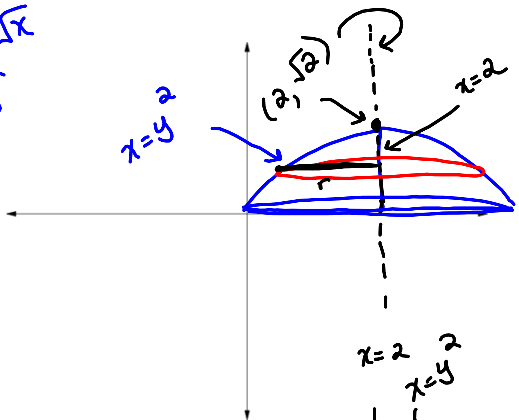
Section 6.2 (continued)

Rotation around lines

4. Sketch the region R bounded by $y = \sqrt{x}$, $y = 0$, $x = 2$. Set up but do not evaluate an integral that finds the volume generated by rotating R about the line $x = 2$.

$y = \sqrt{x}$

$x = y^2$

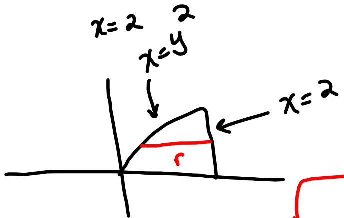


① disk

② variable of integration

$x = 2$ vertical line, integrate with respect to y!

$$V = \int_0^{\sqrt{2}} \pi r^2\, dy$$

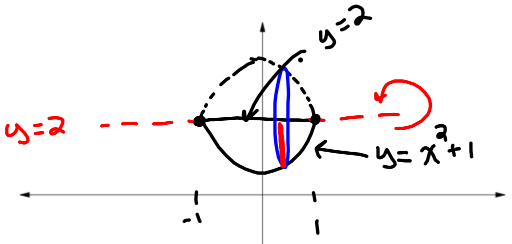


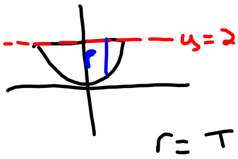
$r =$ right − left

$r = 2 - y^2$

$$\boxed{V = \int_0^{\sqrt{2}} \pi (2 - y^2)^2\, dy}$$

5. Sketch the region R bounded by $y = x^2 + 1$ and $y = 2$. Find the volume generated by rotating R about the line $y = 2$.





① method: disk

② variable of integration: x

$y = 2$ horizontal line

$V = \int_{-1}^{1} \pi r^2 \, dx$ or by symmetry

$V = 2 \int_0^1 \pi r^2 \, dx$

$r = T - B$

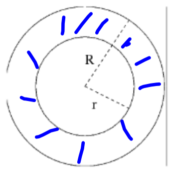
$r = 2 - (x^2 + 1)$

$r = 1 - x^2$

$V = 2\pi \int_0^1 (1 - x^2)^2 \, dx = 2\pi \int_0^1 (1 - 2x^2 + x^4) \, dx = \frac{16\pi}{15}$

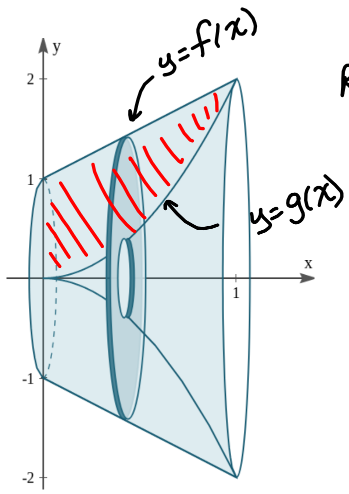
Washer Method: Use when the cross-section of the solid is in the shape of a washer.

Consider the washer shown here:



$\pi R^2 - \pi r^2 = \pi(R^2 - r^2)$

The area of this washer is $\boxed{A = \pi R^2 - \pi r^2 = \pi(R^2 - r^2).}$



R = curve Farthest from axis of rotation

$R = f(x)$

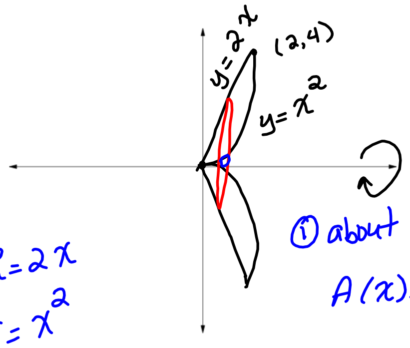
$A(x) = \pi \left(f(x)^2 - g(x)^2 \right)$

r = curve closest to axis of rotation

$r = g(x)$

For the image above, $R = f(x)$ and $r = g(x)$, therefore $V = \int_a^b \pi \left([f(x)]^2 - [g(x)]^2 \right) dx$.

6. Sketch the region R bounded by $y = x^2$, $y = 2x$. Find the volume generated by rotating R about the ~~y-axis.~~ x-axis, then y-axis. find volume



intersection: $x^2 = 2x$

$x^2 - 2x = 0$

$x(x-2) = 0$

$x=0, \quad x=2$

$y=0, \quad y=4$

$R = 2x$

$r = x^2$

① about x-axis:

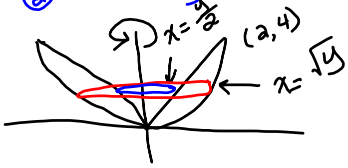
$A(x) = \pi(R^2 - r^2)$

$= \pi(4x^2 - x^4)$

$V = \int_0^2 \pi(4x^2 - x^4)\,dx$

$= \pi\left(\frac{4x^3}{3} - \frac{x^5}{5}\right)\Big|_0^2 = \pi\left(\frac{32}{3} - \frac{32}{5}\right) = \boxed{\frac{64\pi}{15}}$

② about y-axis



$A(y) = \pi\left(\left(\sqrt{y}\right)^2 - \left(\frac{y}{2}\right)^2\right)$

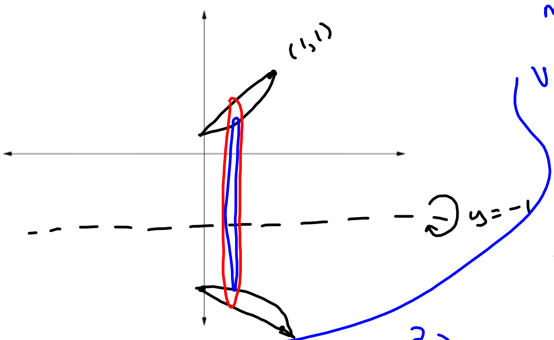
$V = \int_0^4 \pi\left(y - \frac{y^2}{4}\right)dy = \pi\left(\frac{y^2}{2} - \frac{y^3}{12}\right)\Big|_0^4$

$= \pi\left(8 - \frac{64}{12}\right) = \boxed{\frac{8\pi}{3}}$

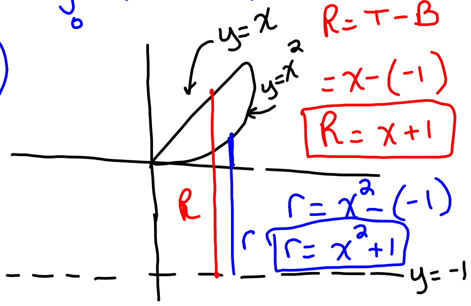
7. Sketch the region R bounded by $y = x$, $y = x^2$. Find the volume generated by rotating R about the line $y = -1$.

variable: x

method: washers



$V = \int_0^1 \pi(R^2 - r^2)\,dx$



$R = T - B$

$= x - (-1)$

$\boxed{R = x+1}$

$r = x^2 - (-1)$

$\boxed{r = x^2+1}$

$V = \pi\int_0^1 \left((x+1)^2 - (x^2+1)^2\right)dx$

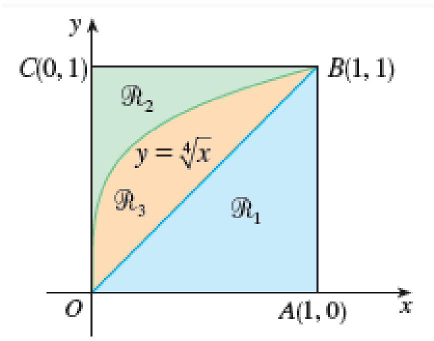
$= \pi\int_0^1 \left(x^2 + 2x + 1 - (x^4 + 2x^2 + 1)\right)dx$

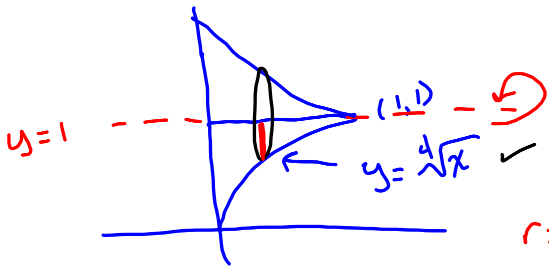
$= \pi\int_0^1 \left(-x^4 - x^2 + 2x\right)dx$

$= \boxed{\frac{7\pi}{15}}$

8. Refer to the figure below to set up but do not evaluate an integral that finds the volume generated by rotating the given region about the spe

(a) R_2 about BC





$r = T - B$

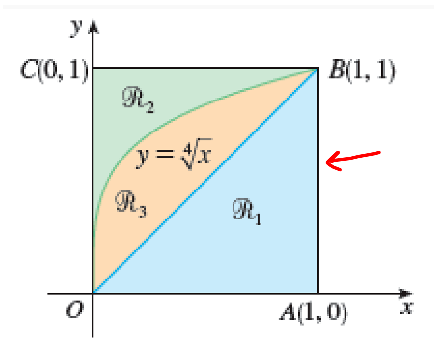
$r = 1 - \sqrt[4]{x}$

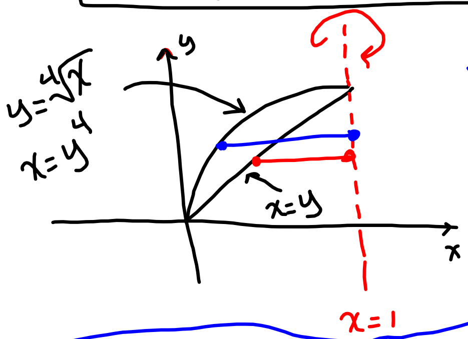
① variable dx

② method disk

$$\boxed{V = \int_0^1 \pi r^2 \, dx = \int_0^1 \pi \left(1 - \sqrt[4]{x}\right)^2 dx}$$

(b) R_3 about AB





vertical line rotation → dy

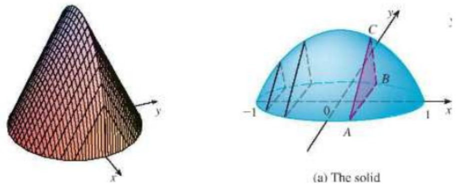
R = right − left

$R = 1 - y^4$

$r = 1 - y$

$$\boxed{V = \int_0^1 \pi \left(\left(1 - y^4\right)^2 - \left(1 - y\right)^2 \right) dy}$$

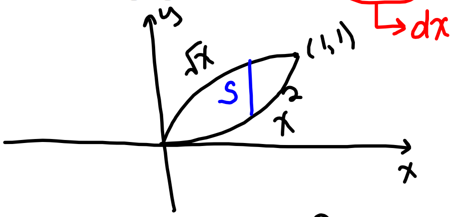
The Method of Slicing: Here, the solid is not the result of a revolution. Rather, the solid is defined by describing the base (bottom) of the solid, and the shape of a cross-section perpendicular to one of the coordinate axes. For example, in the illustration below, the base of the solid is an elliptical region and cross-sections perpendicular to the x-axis are equilateral triangles. To find the volume, integrate the area of a cross-section, $A(x)$ in this case, for $-1 \le x \le 1$, that is $V = \int_{-1}^{1} A(x)\,dx$, where $A(x)$ is the area of the equilateral triangle for $-1 \le x \le 1$.



(a) The solid

9. Find the volume of S where the base of S is the region bounded by $y = x^2$ and $y = \sqrt{x}$. The cross sections perpendicular to the x-axis are squares.

$\to dx$



① variable of integration: dx

② $V = \int_0^1 \left(A_{square}\right) dx$

$A_{square} = S^2$

$S = T - B = \sqrt{x} - x^2$

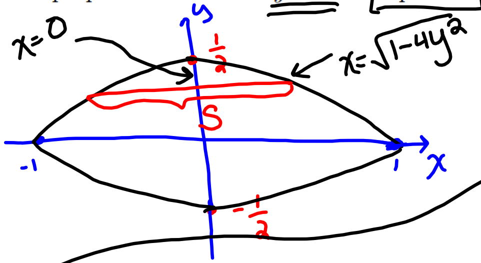
$S^2 = \left(\sqrt{x} - x^2\right)^2$

$\int_0^1 \left(\sqrt{x} - x^2\right)^2 dx$

$\int_0^1 \left(x - 2\sqrt{x}\,x^2 + x^4\right) dx$

$\int_0^1 \left(x - 2x^{\frac{5}{2}} + x^4\right) dx = \boxed{\frac{9}{70}}$

10. Find the volume of the solid S whose base is the ellipse $x^2 + 4y^2 = 1$. The cross sections of S perpendicular to the y-axis are squares.



variable of integration: dy

$V = \int_{-\frac{1}{2}}^{\frac{1}{2}} A_{square}\,dy = 2\int_0^{\frac{1}{2}} A_{square}\,dy$

symmetry

$S =$ right $-$ left

solve $x^2 + 4y^2 = 1$ for x

$x = \pm\sqrt{1-4y^2}$

by symmetry: $S = 2\left[\sqrt{1-4y^2} - 0\right]$

$S = 2\sqrt{1-4y^2}$

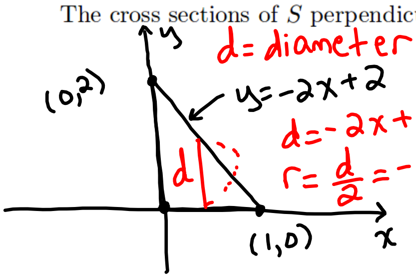
$S^2 = 4(1-4y^2)$

$V = 2\int_0^{\frac{1}{2}} 4(1-4y^2)\,dy$

$V = 8\int_0^{\frac{1}{2}} (1-4y^2)\,dy = \boxed{\frac{8}{3}}$

11. Find the volume of the solid S whose base is the triangular region with vertices $(0,0)$, $(1,0)$ and $(0,2)$. The cross sections of S perpendicular to the x-axis are semi-circles.

d = diameter



$d=-2x+2$

$r=\frac{d}{2}=-x+1$

① variable: dx

② $V=\int_0^1 \left(A_{semicircle}\right)dx$

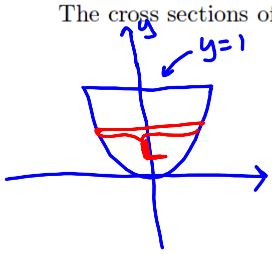
$A_{semicircle} = \frac{1}{2}\pi r^2$

$= \frac{1}{2}\pi(-x+1)^2$

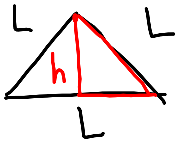
$= \int_0^1 \frac{\pi}{2}(-x+1)^2\,dx$

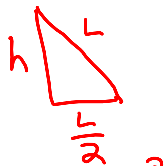
$= \frac{\pi}{2}\int_0^1 (x^2-2x+1)\,dx = \boxed{\frac{\pi}{6}}$

12. Find the volume of the solid S whose base is the region bounded by the parabola $y=x^2$ and $y=1$. The cross sections of S perpendicular to the y-axis are equilateral triangles.



$V=\int_0^1 \left(A_{triangle}\right)dy$





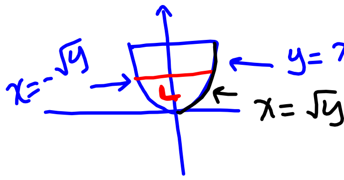
$L^2 = h^2 + \frac{L^2}{4}$

$h = \frac{\sqrt{3}}{2}L$

$A_{equilateral\,tri} = \frac{1}{2}bh$

$= \left(\frac{1}{2}L\right)h$

$= \frac{1}{2}L\left(\frac{\sqrt{3}}{2}L\right)$



$y=x^2$, so $x=\pm\sqrt{y}$

$= \frac{\sqrt{3}}{4}L^2$

$L = R - L = \sqrt{y}-(-\sqrt{y}) = 2\sqrt{y}$

$= \frac{\sqrt{3}}{4}(2\sqrt{y})^2 = \boxed{\sqrt{3}\,y}$

so $V=\int_0^1 \sqrt{3}\,y\,dy = \boxed{\frac{\sqrt{3}}{2}}$