## Section 6.4: Work

In physics, Work is done when a force $F$ that is applied to an object moves that object. Intuitively, we can think of a Force as describing a push or pull on an object. Therefore if we are moving an object, we must apply a force that is equal to the weight of the object. Thus the force $F$ acting on the object is the object's mass times the acceleration due to gravity. More specifically, if an object with mass $m$ moves along a straight path with position function $s(t)$, then the force is given by $F=($ mass of object) x (acceleration due to gravity), that is

$$
F=m \frac{d^{2} s}{d t^{2}}
$$

In the case where the force $F$ is constant, then the Work $W$ done in moving an object a distance $d$ meters is given by

$$
W=F d
$$

Note: Units for Force and Work depend on what metric system is being used:

|  | Force | Distance | Work |
| :---: | :---: | :---: | :---: |
| SI Units | Newtons (N) <br> To convert from mass <br> (in kg) to Newtons, multiply <br> by gravity: 9.8 m/s | Meters (m) | Joules (J) <br> $1 \mathrm{~N} \cdot \mathrm{~m}=1 \mathrm{~J}$ |
| US Units | Pounds (lb) | Feet (ft) | Foot-Pounds (ft-lb) |

Constant force If an object is moving along a straight path with constant force $F$, then the work done in moving the object a distance $d$ is $W=F d$.

1. How much work is done in raising a 60 kg barbell from the floor to a height of 2 meters?
2. How much work is done in lifting a 50 pound weight from a height of 6 inches to a height of 18 inches?

Variable force Suppose an object is moving horizontally from $x=a$ to $x=b$, and for each $x$ between $a$ and $b$, a variable force, $f(x)$, acts on the object. If we partition the interval $[a, b]$ into $n$ sub-intervals with length $\Delta x$, then the work done in moving the object from $x=a$ to $x=b$ can be approximated by $W \approx \sum_{i=1}^{n} f\left(x_{i}\right) \Delta x_{i}$. Therefore the work done in moving an object from $x=a$ to $x=b$ is given by

$$
W=\int_{a}^{b} f(x) d x
$$

3. When a particle is at a distance $x$ meters from the origin, a force of $f(x)=2 x^{2}+1$ Newtons acts on the object, thus causing the object to move horizontally along the $x$-axis. How much work is done in moving the object from $x=1$ to $x=2$ ?

Spring Problems: Hooke's Law says The force $\mathbf{F}$ required to maintain a spring stretched $x$ units beyond its natural length is $F(x)=k x$.


The work $\mathbf{W}$ done in stretching a spring from $x=a$ units beyond its natural length to $x=b$ units beyond its natural length is $W=\int_{a}^{b} f(x) d x$.
4. A spring has a natural length of 1 m . If a $25-\mathrm{N}$ force is required to keep it stretched to a length of 3 m , how much work is done in stretching the spring from 2 m to 5 m ?
5. Suppose the work needed to stretch a spring from its natural length to a length of 5 feet beyond its natural length is $30 \mathrm{ft}-\mathrm{lb}$.
a.) How much work is done in stretching the spring from 3 feet beyond its natural length to 120 inches beyind its natural length?
b.) How far beyond its natural length will a force of 60 lb keep the spring stretched?

Water pumping problems: We want to learn how much work it takes to pump water to the top of a tank. We will do this by finding the work done to pump a 'slice' of water to the top of the tank. In order to do this, we must first determine how much this slice of water weighs, which is the volume, $V_{\text {slice }}$, of the slice (NOT the volume of the tank) times the the weight density of the water (denoted by $\rho g$ ). Thus the force acting on this slice is $F=V_{\text {slice }} * \rho g$. Next, we find the work required to pump this slice to the top by multiplying the force $F$ by the distance from the slice to the top of the tank, that is $W=V_{\text {slice }} * \rho g * d$. Finally, we will integrate to find the total work.
6. A tank is in the shape of cylinder with radius 5 feet and height 20 feet. The tank is full of water to a depth of 12 feet.

a.) Find the work required to pump the water to the top of the tank. Note: The weight density of water is $\rho g=62.5 \mathrm{lbs} / \mathrm{ft}^{3}$.
b.) How much work is done if we only pump the top 3 feet to the top of the tank?
7. A triangular trough has a length of 5 feet, a distance of 2 feet across the top and a height of 3 ft . Assuming it is full of water, set up but do not evaluate an integral that gives the work done in pumping the water through a spout located at the top of the trough with length 0.5 feet. Note: The weight density of water is $\rho g=62.5 \mathrm{lbs} / f t^{3}$.

8. A spherical tank with radius 4 m is half full of a liquid that has density of 900 kg per cubic meter. Find the work done in pumping this liquid through a 1 meter high spout which is located at the top of the sphere.

9. A tank is in the shape of a cone with radius $r=3$ feet and height $h=8$ feet. Assuming it is full of water, set up but do not evaluate an integral that gives the work it takes to pump the the water to the top of the tank. Note: weight density of water is $\rho g=62.5 \mathrm{lbs} / f t^{3}$


## Rope pulling problems:

10. A 100 foot long rope that weighs $\frac{1}{5}$ pounds per foot hangs vertically from the top of a tall building. Find the work done in pulling the entire rope to the top of the building.
11. A 500 foot rope that weighs 3 pounds per foot is used to lift an 80 pound weight up the side of a building that is 625 feet tall. Find the work done.
12. A 200 pound cable is 100 feet long and hangs vertically from the top of a tall building. How much work is done in pulling the first 10 feet of the cable to the top of the building?
