

Section 7.2: Trigonometric Integrals

Identities needed in this section:

1. $\sin^2 x + \cos^2 x = 1$
2. $\tan^2 x + 1 = \sec^2 x$
3. $\cot^2 x + 1 = \csc^2 x$
4. $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$
5. $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$

Type I: Integrals of the form $\int \sin^m x \cos^n x dx$, where m and n are non-negative integers.

Case 1: EITHER m or n (or both) is odd. If the power on $\sin x$ is odd, factor out one $\sin x$ and let $u = \cos x$. If the power on $\cos x$ is odd, factor out one $\cos x$ and let $u = \sin x$. If both powers are odd, do one of the above mentioned but not both.

1. $\int \sin^4 x \cos^3 x dx$ power on cosine is odd.
Factor out one $\cos x$.

$$\begin{aligned} & \int \underbrace{\sin^4 x}_{u^4} \underbrace{\cos^2 x}_{1-\sin^2 x} \underbrace{\cos x dx}_{du} \\ & \qquad\qquad\qquad u = \sin x \\ & \qquad\qquad\qquad du = \cos x dx \\ & \qquad\qquad\qquad \downarrow \\ & \qquad\qquad\qquad 1-u^2 \\ & \int u^4 (1-u^2) du = \int u^4 - u^6 du \\ & \qquad\qquad\qquad = \frac{u^5}{5} - \frac{u^7}{7} + C \\ & \qquad\qquad\qquad = \boxed{\frac{\sin^5 x}{5} - \frac{\sin^7 x}{7} + C} \\ & \text{Power on sine is odd} \rightarrow \text{factor out one sine} \end{aligned}$$

$$\begin{aligned} 2. \int \sin^3(10x) dx &= \int \underbrace{\sin^2(10x)}_{1-\cos^2(10x)} \underbrace{\sin(10x) dx}_{-\frac{1}{10} du} & u = \cos(10x) \\ &\qquad\qquad\qquad -\frac{1}{10} du & du = -10 \sin(10x) dx \\ &\qquad\qquad\qquad \downarrow & -\frac{1}{10} du = \sin(10x) dx \\ &\qquad\qquad\qquad 1-u^2 \\ & -\frac{1}{10} \int (1-u^2) du = -\frac{1}{10} \left(u - \frac{u^3}{3} \right) + C \\ & = -\frac{1}{10} \left(\cos(10x) - \frac{\cos^3(10x)}{3} \right) + C \end{aligned}$$

$$\begin{aligned}
 3. \int \cos^5 x \sqrt{\sin x} dx &= \int \cos^4 x \underbrace{\sqrt{\sin x}}_{\sqrt{u}} \frac{\cos x dx}{du} \quad u = \sin x \\
 &\downarrow \\
 &\left(\cos^2 x\right)^2 \\
 &\downarrow \\
 &\left(1 - \sin^2 x\right)^2 \\
 &\downarrow \\
 &\left(1 - u^2\right)^2 \\
 &\quad \int (1-u^2)^2 \sqrt{u} du \\
 &\quad \int (1-2u^2+u^4) u^{\frac{1}{2}} du \\
 &\quad \int \left(u^{\frac{1}{2}} - 2u^{\frac{5}{2}} + u^{\frac{9}{2}}\right) du \\
 &\quad \frac{2}{3}u^{\frac{3}{2}} - 2 \cdot \frac{2}{7}u^{\frac{7}{2}} + \frac{2}{11}u^{\frac{11}{2}} + C \\
 &\quad \frac{2}{3}(\sin x)^{\frac{3}{2}} - \frac{4}{7}(\sin x)^{\frac{7}{2}} + \frac{2}{11}(\sin x)^{\frac{11}{2}} + C \\
 4. \int_0^{\pi/2} \cos^3 x \sin^3 x dx &= \int_0^{\pi/2} \underbrace{\cos^2 x}_{1 - \sin^2 x} \underbrace{\sin^3 x}_{u^3} \underbrace{\cos x dx}_{du} \quad u = \sin x \quad x = \frac{\pi}{2}, u = 1 \\
 &\downarrow \\
 &\left(1 - u^2\right)^2 \\
 &\quad \int_0^1 (1-u^2)^2 u^3 du \quad \int_0^1 (u^3 - u^5) du \\
 &\quad = \frac{1}{4} - \frac{1}{6} = \boxed{\frac{1}{12}}
 \end{aligned}$$

Case 2: BOTH m and n are even. Use the identities $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$ and $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$.

$$\begin{aligned}
 1. \int_0^{\pi/2} \sin^2 x \cos^2 x dx &= \int_0^{\pi/2} \frac{1}{2}(1 - \cos 2x) \cdot \frac{1}{2}(1 + \cos 2x) dx \\
 &= \frac{1}{4} \int_0^{\pi/2} (1 + \cos 2x - \cos 2x - \cos^2 2x) dx \\
 &= \frac{1}{4} \int_0^{\pi/2} (1 - \cos^2 2x) dx \quad \sin^2 x = \frac{1}{2}(1 - \cos 2x) \\
 &= \frac{1}{4} \int_0^{\pi/2} (\sin^2 2x) dx \\
 &= \frac{1}{4} \int_0^{\pi/2} \frac{1}{2}(1 - \cos 4x) dx \\
 &= \frac{1}{8} \left(x - \frac{1}{4} \sin 4x \right) \Big|_0^{\pi/2} \\
 &= \frac{1}{8} \left(\frac{\pi}{2} - \frac{1}{4} \sin(2\pi) - (0 - \frac{1}{4} \sin 0) \right) \\
 &= \frac{\pi}{16}
 \end{aligned}$$

$$2. \int \frac{\cos^2(\ln x)}{x} dx \quad u = \ln x \\ du = \frac{1}{x} dx$$

$$\begin{aligned} \int \cos^2 u du &= \int \frac{1}{2}(1 + \cos 2u) du \\ &= \frac{1}{2}\left(u + \frac{1}{2}\sin 2u\right) + C \\ &= \frac{1}{2}\left(\ln x + \frac{1}{2}\sin(2\ln x)\right) + C \end{aligned}$$

Type II: Integrals of the form $\int \sec^m x \tan^n x dx$, where m and n are non-negative integers.

Case 1: The power on $\tan x$ is odd. Factor out $\sec x \tan x$ and let $u = \sec x$.

$$u = \sec x$$

$$\begin{aligned} 1. \int \tan^3 x \sec^3 x dx &= \int \underbrace{\tan^2 x}_{\sec^2 x - 1} \underbrace{\sec x}_{u} \underbrace{\sec x \tan x dx}_{du} \\ &\quad \downarrow \\ &\quad \sec^2 x - 1 \\ &\quad \downarrow \\ &\quad u^2 - 1 \\ &= \int (u^2 - 1) u^2 du \quad \rightarrow \int (u^4 - u^2) du \\ &= \frac{u^5}{5} - \frac{u^3}{3} + C \\ &= \boxed{\frac{\sec^5 x}{5} - \frac{\sec^3 x}{3} + C} \end{aligned}$$

$$\begin{aligned} 2. \int \cot^5 x \csc x dx &= \int \underbrace{\cot^4 x}_{(\cot^2 x)^2} \underbrace{\csc x \cot x dx}_{-du} \quad u = \csc x \\ &\quad \downarrow \\ &\quad (\csc^2 x - 1)^2 \\ &\quad \downarrow \\ &\quad (u^2 - 1)^2 \\ &\quad \downarrow \\ &\quad - \int (u^2 - 1)^2 du \\ &\quad - \int (u^4 - 2u^2 + 1) du \\ &\quad - \frac{u^5}{5} + \frac{2u^3}{3} - u + C \\ &\quad - \frac{\csc^5 x}{5} + \frac{2}{3} \csc^3 x - \csc x + C \end{aligned}$$

Case 2: The power on $\sec x$ is even. Factor out $\sec^2 x$ and let $u = \tan x$.

$$1. \int x \tan^4(x^2) \sec^4(x^2) dx$$

$$\text{Let } t = x^2$$

$$dt = 2x dx$$

$$\frac{1}{2} \int \tan^4 t \sec^4 t dt \quad \text{even secant} \rightarrow \text{factor out } \sec^2$$

$$\frac{1}{2} \int \underbrace{\tan^4 t}_{u^4} \underbrace{\sec^2 t}_{\downarrow} \underbrace{\sec^2 t dt}_{du}$$

$$u = \tan t \\ du = \sec^2 t dt$$

$$\tan^2 t + 1 \\ u^2 + 1$$

$$\rightarrow \frac{1}{2} \int (u^6 + u^4) du$$

$$\frac{1}{2} \left(\frac{u^7}{7} + \frac{u^5}{5} \right) + C$$

$$\frac{1}{2} \left(\frac{\tan^7 x^2}{7} + \frac{\tan^5 x^2}{5} \right) + C$$

$$u = \tan t, \\ t = x^2$$

2. $\int \tan^9 x \sec^4 x dx$. Note here, the power on tangent is odd *and* the power on secant is even. Hence treat it as a case 1 *OR* case 2.

method 1: Factor out $\sec^2 x$

$$\int \underbrace{\tan x}_{u^9} \underbrace{\sec^2 x}_{\downarrow} \underbrace{\sec^2 x dx}_{du} \quad \begin{aligned} u &= \tan x \\ du &= \sec^2 x dx \end{aligned}$$

$$\tan^2 x + 1 \\ u^2 + 1$$

$$\int u^9 (u^2 + 1) du = \int u^{11} + u^9 du \\ = \frac{u^{12}}{12} + \frac{u^{10}}{10} + C \\ = \frac{\tan^{12} x}{12} + \frac{\tan^{10} x}{10} + C$$

method 2: Factor $\sec x \tan x$

$$\int \underbrace{\tan^8 x}_{u^8} \underbrace{\sec^3 x}_{\downarrow} \underbrace{\sec x \tan x dx}_{du}$$

$$(\tan^2 x)^4$$

$$(\sec^2 x - 1)^4$$

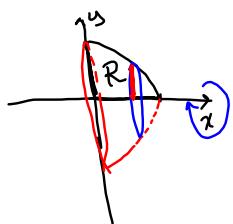
$$(u^2 - 1)^4$$

$$3. \int \tan^2 x dx = \int (\sec^2 x - 1) dx$$

$$= \boxed{\tan x - x + C}$$

Connecting volume of revolution with our new techniques of integration.

1. Find the volume obtained by rotating the region bounded by $y = \cos x$, $y = 0$, $x = 0$ and $x = \frac{\pi}{2}$ about the x axis.



$$\text{disk: } V = \int_0^{\frac{\pi}{2}} \pi r^2 dx \quad r = \cos x - 0$$

$$V = \int_0^{\frac{\pi}{2}} \pi \cos^2 x dx$$

$$= \int_0^{\frac{\pi}{2}} \pi r \cdot \frac{1}{2} (1 + \cos 2x) dx$$

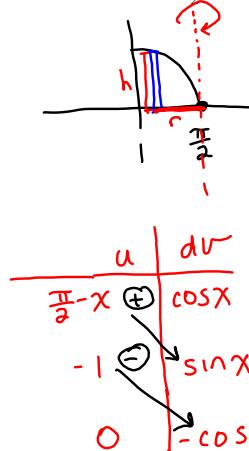
$$= \frac{\pi}{2} \left(x + \frac{1}{2} \sin 2x \right) \Big|_0^{\frac{\pi}{2}}$$

$$= \frac{\pi}{2} \left(\frac{\pi}{2} + \frac{1}{2} \sin \pi - (0 + \frac{1}{2} \sin 0) \right)$$

$$= \boxed{\frac{\pi^3}{4}}$$

2. Find the volume obtained by rotating the region bounded by $y = \cos x$, $y = 0$, $x = 0$ and $x = \frac{\pi}{2}$ about the line $x = \frac{\pi}{2}$

rotation around a vertical line



$$\text{shells } V = \int 2\pi r h dx \quad r = \frac{\pi}{2} - x$$

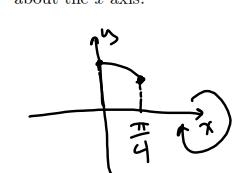
$$h = \cos x$$

$$V = \int_0^{\frac{\pi}{2}} 2\pi (\frac{\pi}{2} - x) \cos x dx$$

$$= 2\pi \left[(\frac{\pi}{2} - x) \sin x - \cos x \right] \Big|_0^{\frac{\pi}{2}}$$

$$= 2\pi (0 - (0 - 1)) = 2\pi$$

3. Find the volume obtained by rotating the region bounded by $y = \sec^2 x$, $y = 0$, $x = 0$ and $x = \frac{\pi}{4}$ about the x axis.



$$V = \int_0^{\frac{\pi}{4}} \pi (\sec^2 x)^2 dx$$

$$= \pi \int_0^{\frac{\pi}{4}} \sec^4 x dx$$

$$u = \tan x \quad x = 0, u = 0 \quad x = \frac{\pi}{4}, u = 1$$

$$du = \sec^2 x dx$$

$$V = \pi \int_0^1 \sec^2 x \sec^2 x du$$

$$= \pi \int_0^1 \tan^2 x + 1 du$$

$$= \pi \int_0^1 (u^2 + 1) du = \pi \left(\frac{u^3}{3} + u \right) \Big|_0^1$$

$$= \boxed{\frac{4\pi}{3}}$$