

### Section 7.3: Trigonometric Substitution

Identities commonly used in this section:

a.)  $\sin^2 \theta + \cos^2 \theta = 1$

b.)  $\tan^2 \theta = \sec^2 \theta - 1$

c.)  $\sin(2x) = 2 \sin x \cos x$

A few interals you must know:

1.  $\int \sec \theta d\theta = \ln |\sec \theta + \tan \theta| + C$

Proof:  $\int \sec \theta d\theta = \int \sec \theta \left( \frac{\sec(\theta) + \tan(\theta)}{\sec(\theta) + \tan(\theta)} \right) d\theta = \int \left( \frac{\sec^2(\theta) + \sec(\theta) \tan(\theta)}{\sec(\theta) + \tan(\theta)} \right) d\theta.$

If we let  $u = \sec(\theta) + \tan(\theta)$ , then  $du = (\sec^2(\theta) + \sec(\theta) \tan(\theta)) d\theta$ , which gives

$$\int \frac{1}{u} du = \ln |u| = \ln |\sec(\theta) + \tan(\theta)|.$$

2.  $\int \sec^3 \theta d\theta = \frac{1}{2} (\sec \theta \tan \theta + \ln |\sec \theta + \tan \theta|) + C$

The general idea of integration by trigonometric substitution is to transform an algebraic integral that involves one of the general forms in the table below into a trig integral that can be integrated using the techniques of section 7.2.

Form	Substitution	Identity Used	Domain
$\sqrt{a^2 - x^2}$	$x = a \sin \theta$	$\cos^2 \theta = 1 - \sin^2 \theta$	$-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$
$\sqrt{a^2 + x^2}$	$x = a \tan \theta$	$\sec^2 \theta = \tan^2 \theta + 1$	$0 \leq \theta \leq \pi, \theta \neq \frac{\pi}{2}$
$\sqrt{x^2 - a^2}$	$x = a \sec \theta$	$\tan^2 \theta = \sec^2 \theta - 1$	$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$$1. \int \frac{dx}{x^2\sqrt{1-x^2}}$$

Form  $a^2 - x^2 \rightarrow x = a \sin \theta$  here,  $a=1$ .

$$\begin{aligned} x &= 1 \cdot \sin \theta \\ dx &= \cos \theta d\theta \end{aligned}$$

$$= \int \frac{\cos \theta d\theta}{\sin^2 \theta \sqrt{1-\sin^2 \theta}}$$

$$= \int \frac{\cos \theta d\theta}{\sin^2 \theta \sqrt{\cos^2 \theta}}$$

$$= \int \frac{\cos \theta}{\sin^2 \theta \cos \theta} d\theta$$

$$\begin{aligned} \sin^2 \theta &= \frac{1}{2}(1-\cos 2\theta) \\ \int \frac{1}{\sin^2 \theta} d\theta &= \int \csc^2 \theta d\theta \end{aligned}$$

$$- \cot \theta + C$$

$$- \frac{A}{O} + C$$

$$\boxed{- \frac{\sqrt{1-x^2}}{x} + C}$$

$$\begin{array}{l} \frac{O}{H} = \sin \theta \\ \frac{1}{1-x^2} \end{array}$$

2.  $\int \frac{\sqrt{x^2+9}}{x^4} dx$  Form  $x^2 + a^2 \rightarrow x = a \tan \theta \quad a=3$

$$\begin{aligned} x &= 3 \tan \theta \\ dx &= 3 \sec^2 \theta d\theta \end{aligned}$$

$$\int \frac{\sqrt{9 \tan^2 \theta + 9}}{81 \tan^4 \theta} 3 \sec^2 \theta d\theta$$

$$\begin{aligned} &\int \frac{\sqrt{9(\tan^2 \theta + 1)}}{81 \tan^4 \theta} 3 \sec^2 \theta d\theta \rightarrow \frac{1}{9} \int \frac{\sec^3 \theta}{\tan^4 \theta} d\theta \\ &\int \frac{\sqrt{9 \sec^2 \theta}}{81 \tan^4 \theta} 3 \sec^2 \theta d\theta \rightarrow \frac{1}{9} \int \frac{1}{\cos^3 \theta} \frac{\cos^4 \theta}{\sin^4 \theta} d\theta \\ &\int \frac{3 \sec \theta 3 \sec^2 \theta d\theta}{81 \tan^4 \theta} \rightarrow \frac{1}{9} \int \frac{\cos \theta}{\sin^4 \theta} d\theta \end{aligned}$$

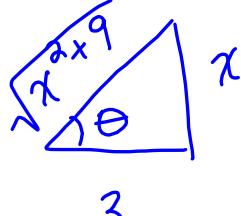
u-sub

$$u = \sin \theta$$

$$du = \cos \theta d\theta$$

recall:

$$x = 3 \tan \theta$$



$$\frac{d}{dx} \frac{x}{3} = \tan \theta$$

$$\sin \theta = \frac{x}{\sqrt{x^2 + 9}}$$

$$= \frac{1}{9} \int u^{-4} du$$

$$= \frac{1}{9} \frac{u^{-3}}{-3} + C$$

$$= -\frac{1}{27 u^3} + C$$

$$= -\frac{1}{27 \sin^3 \theta} + C$$

$$= -\frac{1}{27 \left( \frac{x}{\sqrt{x^2 + 9}} \right)^3} + C$$

3.  $\int \frac{\sqrt{9x^2 - 4}}{x} dx$

Form  $\int \frac{\sqrt{(3x)^2 - a^2}}{x} dx$

$x = a \sec \theta$

$3x = 2 \sec \theta$

$x = \frac{2}{3} \sec \theta$

$dx = \frac{2}{3} \sec \theta \tan \theta d\theta$

$= \int \frac{\sqrt{4 \sec^2 \theta - 4}}{\frac{2}{3} \sec \theta} \frac{1}{3} \sec \theta \tan \theta d\theta$

$= \int \sqrt{4(\sec^2 \theta - 1)} \tan \theta d\theta$

$= \int \sqrt{4 \tan^2 \theta} \tan \theta d\theta$

$= \int 2 \tan^2 \theta d\theta$

$= 2 \int (\sec^2 \theta - 1) d\theta$

$= 2(\tan \theta - \theta) + C$

$= 2 \left( \frac{\sqrt{9x^2 - 4}}{2} - \arcsin \left( \frac{3x}{2} \right) \right) + C$

recall:  $3x = 2 \sec \theta$

$\frac{3x}{2} = \sec \theta$

$$4. \int \frac{x^2}{\sqrt{1-9x^2}} dx = \int \frac{x^2}{\sqrt{1-(3x)^2}} dx \quad a^2-x^2 \rightarrow x = a \sin \theta$$

$$3x = \sin \theta$$

$$x = \frac{1}{3} \sin \theta$$

$$dx = \frac{1}{3} \cos \theta d\theta$$

$$\int \frac{\frac{1}{9} \sin^2 \theta}{\sqrt{1-\sin^2 \theta}} \cdot \frac{1}{3} \cos \theta d\theta$$

$$\frac{1}{27} \int \frac{\sin^2 \theta \cos \theta}{\sqrt{\cos^2 \theta}} d\theta$$

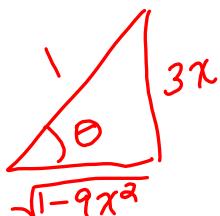
recall:

$$\frac{1}{27} \int \sin^2 \theta d\theta$$

$$\begin{cases} 3x = \sin \theta \\ \text{so } \theta = \arcsin(3x) \end{cases}$$

$$\frac{1}{27} \int \frac{1}{2} (1 - \cos 2\theta) d\theta$$

$$\frac{1}{54} \left( \theta - \frac{1}{2} \sin 2\theta \right) + C$$



$$\frac{1}{54} \left( \theta - \frac{1}{2} (\cancel{2 \sin \theta \cos \theta}) \right)$$

$$\frac{1}{54} \left( \arcsin(3x) - 3x \left( \sqrt{1-9x^2} \right) \right) + C$$

$$5. \int_{5\sqrt{2}}^{10} \frac{dx}{x^3 \sqrt{x^2 - 25}}$$

Form  $x^2 - a^2 \rightarrow x = a \sec \theta$

$$x = 10 \rightarrow 10 = 5 \sec \theta \rightarrow 2 = \sec \theta$$

$$\frac{1}{2} = \cos \theta \rightarrow \theta = \frac{\pi}{3}$$

$$x = 5 \sec \theta$$

$$x = 5\sqrt{2} \rightarrow 5\sqrt{2} = 5 \sec \theta \rightarrow \sqrt{2} = \sec \theta$$

$$\frac{\sqrt{a}}{a} \frac{1}{\sqrt{2}} = \cos \theta$$

$$dx = 5 \sec \theta \tan \theta d\theta$$

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{5 \sec \theta \tan \theta d\theta}{125 \sec^3 \theta \sqrt{25 \sec^2 \theta - 25}}$$

$$\sqrt{25(\sec^2 \theta - 1)}$$

$$\sqrt{25 \tan^2 \theta} = 5 \tan \theta$$

$$\frac{1}{25} \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\tan \theta d\theta}{\sec^2 \theta \cdot 5 \tan \theta} = \frac{1}{125} \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{d\theta}{\sec^2 \theta}$$

$$= \frac{1}{125} \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \cos^2 \theta d\theta$$

$$= \frac{1}{125} \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{1}{2} (1 + \cos 2\theta) d\theta$$

$$= \frac{1}{250} \left( \theta + \frac{1}{2} \sin 2\theta \right) \Big|_{\frac{\pi}{4}}^{\frac{\pi}{3}}$$

$$= \frac{1}{250} \left( \frac{\pi}{3} + \frac{1}{2} \sin \frac{2\pi}{3} - \left( \frac{\pi}{4} + \frac{1}{2} \sin \frac{\pi}{2} \right) \right)$$

$$= \frac{1}{250} \left( \frac{\pi}{3} + \frac{1}{2} \frac{\sqrt{3}}{2} - \frac{\pi}{4} - \frac{1}{2} \right)$$

$$\begin{aligned}
 6. \int_0^{2/3} \frac{1}{(4+9x^2)^{5/2}} dx &= \int_0^{\frac{2}{3}} \frac{dx}{\left(4+(3x)^2\right)^{5/2}}
 \end{aligned}$$

$x = \frac{2}{3} \rightarrow 2 = 2\tan\theta \rightarrow 1 = \tan\theta \quad \boxed{\theta = \frac{\pi}{4}}$   
 $x = 0 \rightarrow 0 = 2\tan\theta \rightarrow 0 = \tan\theta \quad \boxed{\theta = 0}$

$x = \frac{2}{3}\tan\theta$   
 $dx = \frac{2}{3}\sec^2\theta d\theta$

$$\int_0^{\frac{\pi}{4}} \frac{\frac{2}{3}\sec^2\theta d\theta}{\left(\frac{4+4\tan^2\theta}{4\sec^2\theta}\right)^{5/2}}$$

$$\int_0^{\frac{\pi}{4}} \frac{\frac{2}{3}\sec^2\theta d\theta}{(4\sec^2\theta)^{5/2}}$$

$$\frac{\frac{2}{3}}{4} \int_0^{\frac{\pi}{4}} \frac{\sec^2\theta d\theta}{\sec^5\theta} = \frac{1}{6} \int_0^{\frac{\pi}{4}} \frac{d\theta}{\sec^3\theta}$$

$$\frac{1}{3} \int_0^{\frac{\pi}{4}} \frac{d\theta}{32\sec^3\theta} = \frac{1}{96} \int_0^{\frac{\pi}{4}} \cos^3\theta d\theta$$

$\theta = \frac{\pi}{4}, u = \frac{\sqrt{2}}{2}$   
 $\theta = 0, u = 0$   
 $du = \cos\theta d\theta$

$$\frac{1}{96} \int_0^{\frac{\pi}{4}} \frac{\cos^2\theta \cos\theta d\theta}{1-\sin^2\theta} = \frac{1}{96} \int_0^{\frac{\pi}{4}} \frac{\cos^2\theta \cos\theta d\theta}{\cos^2\theta} = \frac{1}{96} \int_0^{\frac{\pi}{4}} \cos\theta d\theta$$

$$\begin{aligned}
 &\frac{1}{96} \int_0^{\frac{\pi}{4}} \cos\theta d\theta \\
 &\left. \frac{1}{96} \left( u - \frac{u^3}{3} \right) \right|_0^{\frac{\pi}{4}} = \frac{1}{96} \left( \frac{\sqrt{2}}{2} - \frac{1}{3} \left( \frac{\sqrt{2}}{2} \right)^3 \right)
 \end{aligned}$$

$$7. \int \frac{dx}{\sqrt{x^2 - 8x}}$$

complete square:

$$\underbrace{x^2 - 8x + 16 - 16}_{=}$$

$$\left(\frac{-8}{2}\right)^2 = 16$$

$$(x-4)^2 - 16$$

$$\int \frac{dx}{\sqrt{(x-4)^2 - 16}}$$

$$x-4 = 4 \sec \theta$$

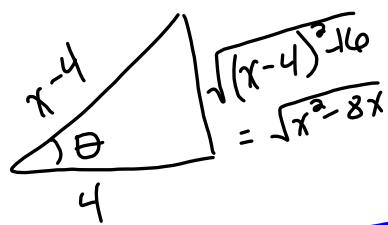
$$dx = 4 \sec \theta \tan \theta d\theta$$

$$\int \frac{4 \sec \theta \tan \theta d\theta}{\sqrt{16 \sec^2 \theta - 16}}$$

$$\sqrt{16(\sec^2 \theta - 1)} = \sqrt{16 \tan^2 \theta} = 4 \tan \theta$$

Formula page!  
on cover page!

$$\int \sec \theta d\theta = \ln |\sec \theta + \tan \theta| + C$$



$$\text{recall } x-4 = 4 \sec \theta$$

$$\frac{H}{A} \frac{x-4}{4} = \sec \theta$$

$$= \boxed{\ln \left| \frac{x-4}{4} + \frac{\sqrt{x^2 - 8x}}{4} \right| + C}$$

## Section 7.3 (continued)

$$8. \int \frac{1}{(x^2 + 6x + 13)^{3/2}} dx$$

$$\underbrace{x^2 + 6x + 9}_{(x+3)^2} + \underbrace{13 - 9}_4$$

$$\left(\frac{6}{2}\right)^2 = 9$$

$$\int \frac{dx}{\left[(x+3)^2 + 4\right]^{3/2}}$$

$$x+3 = 2\tan\theta$$

$$dx = 2\sec^2\theta d\theta$$

$$4^{\frac{3}{2}} = 2^3 = 8$$

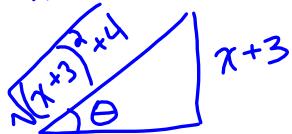
$$\int \frac{2\sec^2\theta d\theta}{\left[4\tan^2\theta + 4\right]^{3/2}} = \int \frac{2\sec^2\theta d\theta}{(4\sec^2\theta)^{3/2}}$$

$$= \int \frac{2\sec^2\theta d\theta}{8\sec^3\theta}$$

recall:  $x+3 = 2\tan\theta$

$$= \frac{1}{4} \int \frac{d\theta}{\sec\theta}$$

$$\frac{\theta}{A} \quad \frac{x+3}{2} = \tan\theta$$



2

$$= \frac{1}{4} \int \cos\theta d\theta$$

$$= \frac{1}{4} \sin\theta + C$$

$$= \frac{1}{4} \left( \frac{x+3}{\sqrt{x^2 + 6x + 13}} \right) + C$$

