

### Section 7.4: Integration by Partial Fractions

Suppose  $f(x) = \frac{g(x)}{h(x)}$  where  $g(x)$  and  $h(x)$  are polynomials *and* the degree of  $h(x)$  is **greater** than the degree of  $g(x)$  (if this is not true, you must first do long division). In order to integrate a partial fraction problem, you must first find the Partial Fraction Decomposition:

- First **FACTOR** the denominator BEFORE setting up the decomposition

- **Case I:** The denominator is a product of *linear* factors, *none* repeating, then the Partial Fraction Decomposition has the form:

$$\frac{x+1}{(x-2)(2x-11)} = \frac{A}{x-2} + \frac{B}{2x-11}$$

- **Case II:** The denominator is a product of *linear* factors, *some* repeating, then the Partial Fraction Decomposition has the form:

$$\frac{x+1}{(x-1)(x-2)^3} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{(x-2)^2} + \frac{D}{(x-2)^3}$$

- **Case III:** The denominator contains *irreducible quadratic* factors, *none* repeating, then the Partial Fraction Decomposition has the form:

$$\frac{x+1}{(x-2)^2(x^2+1)} = \frac{A}{x-2} + \frac{B}{(x-2)^2} + \frac{Cx+D}{x^2+1}$$

- **Case IV:** The denominator contains *irreducible quadratic* factors, *some* repeating, then the Partial Fraction Decomposition has the form:

$$\frac{x+5}{(x-2)(x^2+4)^2} = \frac{A}{x-2} + \frac{Bx+C}{x^2+4} + \frac{Dx+E}{(x^2+4)^2}$$

Once you find  $A$ ,  $B$ , etc., then you integrate the result.

A common integral:  $\int \frac{1}{x^2+a^2} dx = \frac{1}{a} \arctan\left(\frac{x}{a}\right) + C$

$$1. \int \frac{1}{2x^3 + 5x^2 + 3x} dx = \textcircled{1} \text{ Higher power on bottom } \leftarrow$$

\textcircled{2} Factor denominator

$$\frac{1}{x(2x^2 + 5x + 3)} = \frac{1}{x(x+1)(2x+3)}$$

$$\text{PFD : } \frac{1}{x(x+1)(2x+3)} = \left( \frac{A}{x} + \frac{B}{2x+3} + \frac{C}{x+1} \right) \frac{1}{x(x+1)(2x+3)}$$

$$1 = A(2x+3)(x+1) + Bx(x+1) + Cx(2x+3)$$

$$x=0 : 1 = A(3) \rightarrow A = \frac{1}{3}$$

$$x = -\frac{3}{2} : 1 = B\left(-\frac{3}{2}\right)\left(-\frac{1}{2}\right) \rightarrow 1 = \frac{3}{4}B \rightarrow B = \frac{4}{3}$$

$$x = -1 : 1 = C(-1)(1) \rightarrow C = -1$$

$$\int \left( \frac{\frac{1}{3}}{x} + \frac{\frac{4}{3}}{2x+3} - \frac{1}{x+1} \right) dx = \boxed{\frac{\frac{1}{3} \ln|x| + \frac{4}{3} \frac{1}{2} \ln|2x+3| - \ln|x+1|}{+ C}}$$

$\begin{aligned} u &= 2x+3 \\ du &= 2dx \\ \frac{1}{2} \int \frac{du}{u} &= dx \end{aligned}$ 
  
 $\begin{aligned} u &= x+1 \\ du &= dx \\ \int \frac{1}{u} du &= dx \end{aligned}$

$$2. \int_{-1}^0 \frac{x^3}{x+2} dx = \textcircled{1} \text{ power on bottom is } \underline{\text{not}} \text{ larger than power on top} \rightarrow \text{LONG DIVISION!}$$

$$\begin{array}{r} Q \\ \overline{x+2 \longdiv{x^3 - 2x^2 + 4}} \\ \underline{-x^3 - 2x^2} \\ \underline{\underline{-2x^2}} \end{array}$$

$$Q + \frac{R}{D}$$

$$\begin{array}{r} +2x^2 + 4x \\ \underline{4x} \\ \underline{-4x - 8} \\ -8 \end{array}$$

$$\int_{-1}^0 \left( x^2 - 2x + 4 + \frac{-8}{x+2} \right) dx$$

$$\left. \left( \frac{x^3}{3} - x^2 + 4x - 8 \ln|x+2| \right) \right|_{-1}^0$$

$$-8 \ln 2 - \left( -\frac{1}{3} - 1 - 4 \right)$$

$$-8 \ln 2 + \frac{16}{3}$$

3.  $\int \frac{1}{(x-1)^2(x+4)} dx =$  ① higher power on bottom  
 $(x-1)^2$  repeating linear factor!

PFD:

$$\frac{1}{(x-1)^2(x+4)} = \left( \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+4} \right) \frac{1}{(x-1)^2(x+4)}$$

$$1 = A(x-1)(x+4) + B(x+4) + C(x-1)^2$$

$$x=1: 1 = B(5) \rightarrow B = \frac{1}{5}$$

$$x=-4: 1 = C(-5)^2 \rightarrow C = \frac{1}{25}$$

$$\rightarrow 1 = A(x-1)(x+4) + \frac{1}{5}(x+4) + \frac{1}{25}(x-1)^2$$

$$x=0: 1 = A(-4) + \frac{4}{5} + \frac{1}{25}$$

$$1 - \frac{4}{5} - \frac{1}{25} = -4A \rightarrow \frac{4}{25} = -4A$$

$$\frac{25 - 20 - 1}{25} = -4A \rightarrow A = -\frac{1}{25}$$

$$\int \left( \frac{-1/25}{x-1} + \frac{1/5}{(x-1)^2} + \frac{1/25}{x+4} \right) dx$$

$$\begin{aligned} u &= x-1 \\ du &= dx \\ \int \frac{1}{u^2} du &= -\frac{1}{u} \end{aligned}$$

$$\boxed{\frac{-1}{25} \ln|x-1| + \frac{1}{5} \left( -\frac{1}{x-1} \right) + \frac{1}{25} \ln|x+4| + C}$$

4.  $\int \frac{x^4 - 11x^2 + 14x + 8}{x^3 - 4x^2 + 4x} dx =$  ① degree on top is higher!  
Long division!

$$\begin{array}{r} \text{D} \quad x+4 \text{ a} \\ x^3 - 4x^2 + 4x \quad | \quad x^4 + 0x^3 - 11x^2 + 14x + 8 \\ \underline{-x^4 + 4x^3 + 4x^2} \\ 4x^3 - 15x^2 + 14x + 8 \\ \underline{-4x^3 + 16x^2 + 16x} \\ x^2 - 2x + 8 \quad R \end{array}$$

$$\int \left( x+4 + \frac{x^2 - 2x + 8}{x^3 - 4x^2 + 4x} \right) dx = \int \left( x+4 + \frac{x^2 - 2x + 8}{x(x-2)^2} \right) dx$$

PF D

$$\frac{x^2 - 2x + 8}{x(x-2)^2} = \left( \frac{A}{x} + \frac{B}{(x-2)} + \frac{C}{(x-2)^2} \right) x(x-2)^2$$

$$x^2 - 2x + 8 = A^2(x-2)^2 + Bx(x-2) + Cx^4$$

$$x=0: \quad 8 = A(4) \rightarrow A=2$$

$$x=2: \quad 8 = C(2) \rightarrow C=4$$

$$x^2 - 2x + 8 = 2(x-2)^2 + Bx(x-2) + 4x$$

$$x=1: \quad 7 = 2(1) + B(-1) + 4$$

$$7 = 6 - B \rightarrow B = -1$$

$$\int \left( x+4 + \frac{2}{x} + \frac{-1}{x-2} + \frac{4}{(x-2)^2} \right) dx$$

$$\boxed{\frac{x^2}{2} + 4x + 2 \ln|x| - \ln|x-2| - \frac{4}{x-2} + C}$$

5.  $\int \frac{3x^3 + 18}{x^2(x^2 + 9)} dx =$

$x^2 + 9 = \text{irreducible quadratic !!!!}$   
 $x^2 = \text{repeating linear!}$

PFD :  $\frac{3x^3 + 18}{x^2(x^2 + 9)} = \left( \frac{A}{x} + \frac{B}{x^2} + \frac{Cx + D}{x^2 + 9} \right) \frac{2}{x^2(x^2 + 9)}$

$3x^3 + 18 = A x(x^2 + 9) + B(x^2 + 9) + (Cx + D)x^2$

$x=0 :$   $18 = B(9)$   $B = 2$   $\int \left( \frac{2}{x^2} + \frac{3x - 2}{x^2 + 9} \right) dx$

$3x^3 + 18 = A x(x^2 + 9) + 2(x^2 + 9) + (Cx + D)x^2$

$= \underline{A} x^3 + \underline{9A} x + \underline{2} x^2 + \underline{18} + \underline{Cx^3} + \underline{dx^2}$

$3x^3 + 0x^2 + 0x + 18 = (\underline{A} + \underline{C})x^3 + (\underline{2} + \underline{D})x^2 + \underline{9A}x + \underline{18}$

$3 = A + C \rightarrow C = 3$   
 $0 = 2 + D \rightarrow D = -2$   
 $0 = 9A \rightarrow A = 0$

$18 = 18$

$\int \left( \frac{2}{x^2} + \frac{3x - 2}{x^2 + 9} \right) dx$

Recall

 $\int \frac{1}{x^2 + a^2} dx = \frac{1}{a} \arctan \frac{x}{a}$

$\int \left( \frac{2}{x^2} + \frac{3x}{x^2 + 9} - \frac{2}{x^2 + 9} \right) dx$ 

u-sub  
u =  $x^2 + 9$

$a = 3$

$-\frac{2}{x} + \frac{3}{2} \ln|x^2 + 9| - 2 \cdot \frac{1}{3} \arctan \frac{x}{3} + C$

6.  $\int_0^1 \frac{x+6}{(x^2+1)(x^2+4)} dx =$   $x^2+1$  and  $x^2+4$   
are irreducible quadratics

PF D  $\frac{x+6}{(x^2+1)(x^2+4)} = \left( \frac{Ax+B}{x^2+1} + \frac{Cx+d}{x^2+4} \right) (x^2+1)(x^2+4)$

$$\begin{aligned} x+6 &= (Ax+B)(x^2+4) + (Cx+d)(x^2+1) \\ &= Ax^3 + 4Ax + Bx^2 + 4B + Cx^3 + Cx + dx^2 + d \end{aligned}$$

$$0x^3 + 0x^2 + x + 6 = (A+C)x^3 + (B+d)x^2 + (4A+C)x + 4B + d$$

$$\begin{aligned} 0 &= A+C \rightarrow C = -A \\ 0 &= B+d \rightarrow d = -B \\ 1 &= 4A+C \rightarrow 1 = 4A - A \rightarrow 1 = 3A \\ 6 &= 4B+d \rightarrow 6 = 4B - B \rightarrow 6 = 3B \\ &\quad \boxed{B=2} \\ &\quad \boxed{d=-2} \\ &\quad \boxed{A=\frac{1}{3}} \\ &\quad \boxed{C=-\frac{1}{3}} \end{aligned}$$

$$\int_0^1 \left( \frac{\frac{1}{3}x+2}{x^2+1} + \frac{-\frac{1}{3}x-2}{x^2+4} \right) dx$$

$$\int_0^1 \left( \frac{\frac{1}{3}x}{x^2+1} + \frac{2}{x^2+1} + \frac{-\frac{1}{3}x}{x^2+4} + \frac{-2}{x^2+4} \right) dx$$

$\frac{u-\text{sub}}{u=x^2+1} \quad \arctan x \quad \frac{u-\text{sub}}{u=x^2+4} \quad \frac{1}{2} \arctan \frac{x}{2}$

$$\left. \left( \frac{1}{6} \ln(x^2+1) + 2 \arctan x - \frac{1}{6} \ln(x^2+4) - \arctan \frac{x}{2} \right) \right|_0^1$$

$$\frac{1}{6} \ln 2 + 2 \arctan \frac{\pi}{4} - \frac{1}{6} \ln 5 - \arctan \frac{1}{2} - \left( -\frac{1}{6} \ln 4 \right)$$