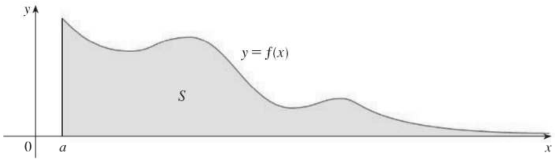
Section 7.8: Improper Integrals

Type I: An Improper Integral is of Type I if one (or both) of the limits of integration is infinite. Here, we study the area under the graph of $f(x)$ for very large (in absolute value) x.



1. $\displaystyle\int_a^\infty f(x)\,dx = \lim_{t\to\infty}\int_a^t f(x)\,dx$
2. $\displaystyle\int_{-\infty}^a f(x)\,dx = \lim_{t\to-\infty}\int_t^a f(x)\,dx$
3. $\displaystyle\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^a f(x)\,dx + \int_a^\infty f(x)\,dx$. If *either* integrals diverge, then $\displaystyle\int_{-\infty}^\infty f(x)\,dx$ diverges. If *both* integrals converge, then $\displaystyle\int_{-\infty}^\infty f(x)\,dx$ converges to the sum of the two integrals.

1. $\displaystyle\int_1^\infty \frac{1}{\sqrt{x}}\,dx$

2. $\displaystyle\int_3^\infty \frac{1}{x}\,dx$

3. $\displaystyle\int_1^\infty \frac{dx}{x^2}$

Note: $\displaystyle\int_1^\infty \frac{1}{x^p}\,dx$ will converge if $p > 1$ and will diverge if $p \le 1$. Also note the lower limit of integration does not have to be 1 for this to be true.

4. $\int_e^\infty \frac{dx}{x \ln x}$

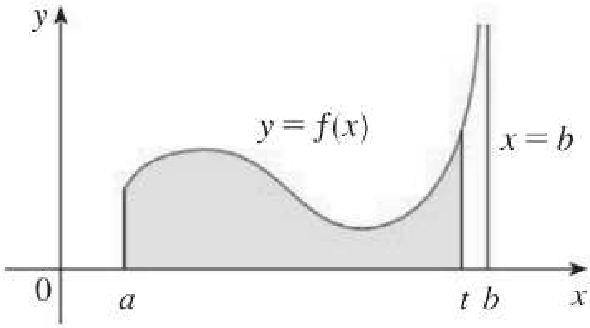
5. $\int_0^\infty x^2 e^{-x^3}\, dx$

6. $\displaystyle\int_6^\infty \frac{1}{x^2 - 7x + 12}\,dx$

7. $\displaystyle\int_{-\infty}^{0} xe^{x}\,dx$

8. $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{x^2+4}$

Type II: An Improper Integral is of Type II if there is a discontinuity (division by 0) on the interval $[a, b]$.



(a) $\displaystyle\int_a^b f(x)\,dx = \lim_{t\to a^+} \int_t^b f(x)\,dx$

(b) Suppose $f(x)$ is discontinuous at $x = b$: Then
$\displaystyle\int_a^b f(x)\,dx = \lim_{t\to b^-} \int_a^t f(x)\,dx$

(c) If $f(x)$ is discontinuous at some c where $a < c < b$, then
$\displaystyle\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$. If *either* of the integrals diverges, then $\displaystyle\int_a^b f(x)\,dx$ diverges.
If *both* integrals converge, then $\displaystyle\int_a^b f(x)\,dx$ converges to the sum of the two integrals.

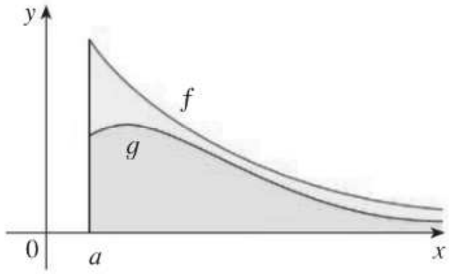
9. $\displaystyle\int_9^{36} \frac{1}{\sqrt[3]{x-9}}\,dx$

10. $\displaystyle\int_0^{5/4} \frac{1}{4x-5}\,dx$

11. $\displaystyle\int_{-1}^{8} \frac{1}{\sqrt[3]{x}}\,dx$

12. $\displaystyle\int_{-1}^{2} \frac{1}{x^4}\,dx$

Comparison Theorem for Improper Integrals:



Suppose $f(x)$ and $g(x)$ are continuous, positive functions on the interval $[a, \infty)$. Also, suppose that $f(x) \geq g(x)$ on the interval $[a, \infty)$. Then:

(i) If $\int_a^\infty f(x)\,dx$ converges, so does $\int_a^\infty g(x)\,dx$. '**If the bigger converges, so does the smaller**'

(**Note:** If $\int_a^\infty f(x)\,dx$ diverges, no conclusion can be drawn about $\int_a^\infty g(x)\,dx$.

(ii) If $\int_a^\infty g(x)\,dx$ diverges, so does $\int_a^\infty f(x)\,dx$. '**If the smaller diverges, so does the larger**'

(**Note:** If $\int_a^\infty g(x)\,dx$ converges, no conclusion can be drawn about $\int_a^\infty f(x)\,dx$).

13. Determine whether the following integrals converge or diverge:

a.) $\int_1^\infty \frac{x}{x^5+1}\,dx$

b.) $\int_2^\infty \frac{1}{x+e^{2x}}\,dx$

c.) $\displaystyle\int_1^{\infty} \frac{\sin^2 x}{x^4}\, dx$

d.) $\displaystyle\int_1^{\infty} \frac{2 + e^{-x}}{x}\, dx$