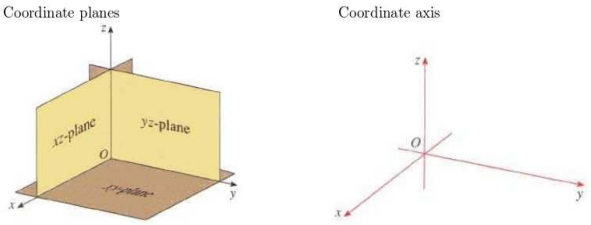
Section 12.1: Three-Dimensional Coordinate System

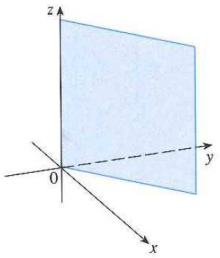
In three dimensions, a point P is represented as an ordered triple $P(x, y, z)$. The orientation of the x, y, and z axes is shown below, and the three axes divide space into eight octants. The first octant, in the foreground, is determined by the positive axes.

We can project a point onto each of the three coordinate planes, for example the projection of the point $(1, 6, 8)$ onto the xy-plane is $(1, 6, 0)$ and likewise the projection of the point $(1, 6, 8)$ onto the yz-plane is $(0, 6, 8)$ and onto the xz-plane is $(1, 0, 8)$.



Example 1: Describe the intersection of the planes $z = 3$ and $y = -2$.

Planes: We know an equation of the form $y = mx + b$ determines a line in $\mathbb{R}^2$. What does it represent in $\mathbb{R}^3$?



An equation of a **plane** is an equation of the form $ax + by + cz = d$.

Example 2: Sketch the graph of $2x + 4y + 6z = 12$.

Cylinders: Recall the equation of a circle in $\mathbb{R}^2$ with center (h, k) and radius r is

$(x-h)^2 + (y-k)^2 = r^2$. What does $(x-h)^2 + (y-k)^2 = r^2$ represent in $\mathbb{R}^3$?

Example 3: Sketch the graph of

a.) $x^2 + y^2 = 4$ b.) $x^2 + z^2 = 9$, $-2 \leq y \leq 5$ c.) $y^2 + z^2 = 4$, $x \geq 0$

Spheres: An equation of a **sphere** with center (h, k, l) and radius r is

$$(x-h)^2 + (y-k)^2 + (z-l)^2 = r^2$$

In what way does a sphere intersect a plane?

(i) The sphere does not intersect the plane. This happens if the distance from the center of the sphere to the plane is *greater than* the radius of the sphere.

(ii) The intersection of a sphere and a plane is a point. This happens if the distance from the center of the sphere to the plane is *equal to* the radius of the sphere. In this case the plane is *tangent* to the sphere.

(iii) The intersection of a sphere and a plane is a circle. This happens if the distance from the center of the sphere to the plane is *less than* the radius of the sphere.

Example 3: Find the center and radius of the sphere $x^2 + y^2 + z^2 + 8x - 6y + 2z + 17 = 0$.

Example 4: What is the equation of the sphere with center $(1, 2, 3)$ that touches the xy- plane?

Example 5: Find the equation of the sphere with radius 3 and center $(1, 4, 3)$. What is the intersection of this sphere with the three coordinate planes?

a.) Intersection with the xy-plane b.) Intersection with the xz-plane c.) Intersection with the yz-plane

Distance formula in three dimensions: The distance between the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ is $|PQ| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$.

Midpoint of a line segment: The midpoint of the line segment joining the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ is $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$.

Example 6: Given the points $A(5, 5, 1)$, $B(3, 3, 2)$, and $C(1, 4, 4)$, determine whether ΔABC is isosceles, right, or both.

Example 7: What is the equation of the sphere if one of its diameters has endpoints $(2, 4, 3)$ and $(1, -6, 4)$?

Example 8: Describe the following regions in the dimension indicated.

a.) $y = 2 - x$ in $\mathbb{R}^2$ and $\mathbb{R}^3$

b.) $x^2 + y^2 \leq 25$ in $\mathbb{R}^2$ and $\mathbb{R}^3$

c.) $1 \leq z \leq 3$ in $\mathbb{R}^3$

d.) $1 < x^2 + y^2 + z^2 < 25$ in $\mathbb{R}^3$