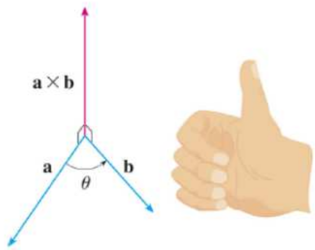
©Amy Austin, January 12, 2020

Section 12.4: The Cross Product

The **cross product** $\overrightarrow{a \times b}$ of two vectors $\vec{a}$ and $\vec{b}$ is a vector, unlike the dot product, which is a scalar. We will see later that $\overrightarrow{a \times b}$ is useful because it is perpendicular to both $\vec{a}$ and $\vec{b}$.



We can find $\overrightarrow{a \times b}$ in component form by using the fact that $\overrightarrow{a \times b}$ is perpendicular to both $\vec{a}$ and $\vec{b}$, along with the vector distributive laws (see page 668-669 in text for proof).

$$\overrightarrow{a \times b} = \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1 \rangle$$

In order to make this expression for $\overrightarrow{a \times b}$ easier to remember, we will use the notation of determinates. A determinate of order 2 is defined by

$$\begin{vmatrix} \text{a} & \text{b} \\ \text{c} & \text{d} \end{vmatrix} = ad - bc$$

A determinate of order 3 is defined by

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

Thus, if we make the first row **i**, **j**, and **k**, the second row **a**, and the third row **b**,

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \mathbf{i} \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \mathbf{j} \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \mathbf{k} \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

$$= \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1 \rangle$$

Example 1: Find the cross product of $\langle 1, 1, 3 \rangle$ and $\langle -2, -1, -5 \rangle$.

$$\begin{vmatrix} i & j & k \\ 1 & 1 & 3 \\ -2 & -1 & -5 \end{vmatrix}$$

Example 2: Find a unit vector perpendicular to both $\langle 1, 2, 1\rangle$ and $\langle 0, 1, 3\rangle$.

Example 3: Find a vector perpendicular to the plane through the points $P(2, 3, 5)$, $Q(-1, 3, 4)$ and $R(3, 0, 6)$.

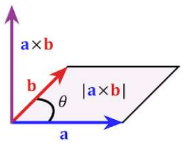
Theorem: If θ is the angle between $\overrightarrow{a}$ and $\overrightarrow{b}$, then $|\overrightarrow{a \times b}| = |\overrightarrow{a}||\overrightarrow{b}|\sin\theta$.

Corollary: Two non zero vectors $\overrightarrow{a}$ and $\overrightarrow{b}$ are parallel if and only if $|\overrightarrow{a \times b}| = 0$.

Example 4: Find the angle between the vectors $\overrightarrow{a} = 2\overrightarrow{i} + 6\overrightarrow{j} - 4\overrightarrow{k}$ and $\overrightarrow{b} = -3\overrightarrow{i} - 9\overrightarrow{j} + 6\overrightarrow{k}$.

Note: we can always find θ using the formula $\overrightarrow{a} \cdot \overrightarrow{b} = |\overrightarrow{a}||\overrightarrow{b}|\cos\theta$.

Theorem: The length of the cross product $\overrightarrow{a} \times \overrightarrow{b}$ is equal to the area of the parallelogram determined by **a** and **b**.



Example 5: Find the area of the triangle determined by the points $P(1,0,0)$, $Q(0,2,0)$ and $R(0,0,3)$.

Example 6: Determine whether each expression is meaningful or meaningless (circle one). If so, state whether the expression is a vector or a scalar.

a.) $\mathbf{a} \cdot \mathbf{b}$ meaningful (vector or scalar) meaningless

b.) $\mathbf{a} \times \mathbf{b}$ meaningful (vector or scalar) meaningless

c.) $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ meaningful (vector or scalar) meaningless

d.) $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ meaningful (vector or scalar) meaningless

e.) $(\mathbf{a} \cdot \mathbf{b}) \times (\mathbf{c} \cdot \mathbf{d})$ meaningful (vector or scalar) meaningless

f.) $(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d})$ meaningful (vector or scalar) meaningless

g.) $\mathbf{a} \times (\mathbf{b} \cdot \mathbf{c})$ meaningful (vector or scalar) meaningless

h.) $|\mathbf{a}|(\mathbf{b} \times \mathbf{c})$ meaningful (vector or scalar) meaningless