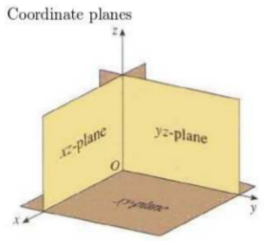
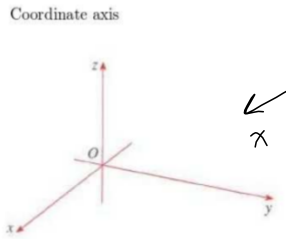
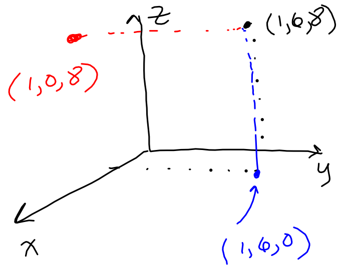
Section 12.1: Three-Dimensional Coordinate System

In three dimensions, a point P is represented as an ordered triple $P(x, y, z)$. The orientation of the x, y, and z axes is shown below, and the three axes divide space into eight octants. The first octant, in the foreground, is determined by the positive axes.

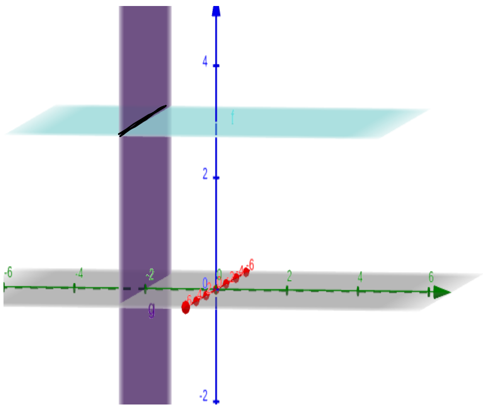
We can project a point onto each of the three coordinate planes, for example the projection of the point $(1, 6, 8)$ onto the xy-plane is $(1, 6, 0)$ and likewise the projection of the point $(1, 6, 8)$ onto the yz-plane is $(0, 6, 8)$ and onto the xz-plane is $(1, 0, 8)$.





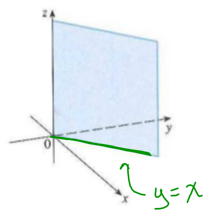


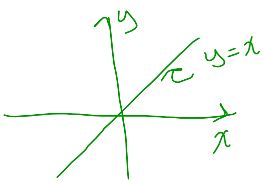
Example 1: Describe the intersection of the planes $z = 3$ and $y = -2$.



Line

Planes: We know an equation of the form $y = mx + b$ determines a line in $\mathbb{R}^2$. What does it represent in $\mathbb{R}^3$?

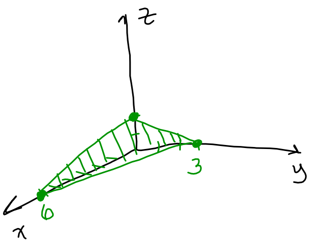




all degree one!

An equation of a **plane** is an equation of the form $ax + by + cz = d$.

Example 2: Sketch the graph of $2x + 4y + 6z = 12$.



x-intercept: $y=0,\ z=0$; $2x=12$; $x=6$

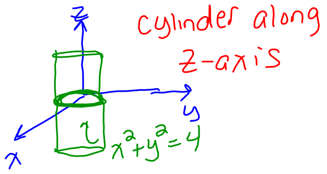
z-int: $6z=12$; $z=2$

y-intercept: $x=0,\ z=0$; $4y=12$; $y=3$

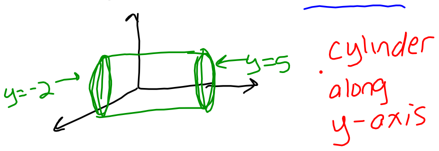
Cylinders: Recall the equation of a circle in $\mathbb{R}^2$ with center (h,k) and radius r is $(x-h)^2+(y-k)^2=r^2$. What does $(x-h)^2+(y-k)^2=r^2$ represent in $\mathbb{R}^3$?

Example 3: Sketch the graph of

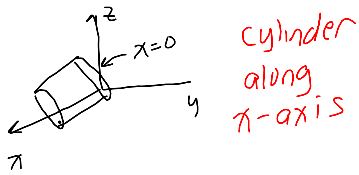
a.) $x^2+y^2=4$



b.) $x^2+z^2=9,\ -2\le y\le 5$



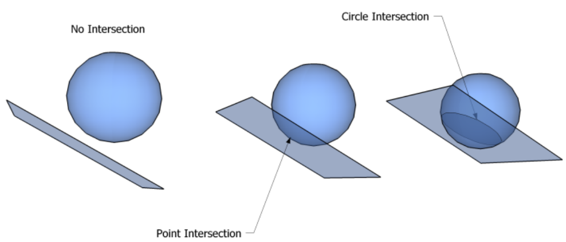
c.) $y^2+z^2=4,\ x\ge 0$



Spheres: An equation of a **sphere** with center (h,k,l) and radius r is

$$(x-h)^2+(y-k)^2+(z-l)^2=r^2$$

In what way does a sphere intersect a plane?



(i) The sphere does not intersect the plane. This happens if the distance from the center of the sphere to the plane is *greater than* the radius of the sphere.

(ii) The intersection of a sphere and a plane is a point. This happens if the distance from the center of the sphere to the plane is *equal to* the radius of the sphere. In this case the plane is *tangent* to the sphere.

(iii) The intersection of a sphere and a plane is a circle. This happens if the distance from the center of the sphere to the plane is *less than* the radius of the sphere.

Example 3: Find the center and radius of the sphere $x^2 + y^2 + z^2 + 8x - 6y + 2z + 17 = 0$.

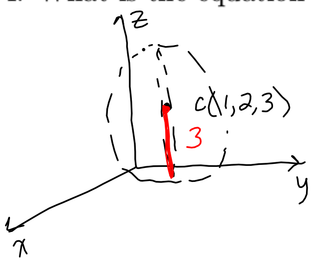
$x^2 + 8x + 16 \quad y^2 - 6y + 9 \quad + z^2 + 2z + 1 \quad = -17 + 16 + 9 + 1$

$(x+4)^2 + (y-3)^2 + (z+1)^2 = 9$

$C = (-4, 3, -1), \quad r = 3$

Example 4: What is the equation of the sphere with center $(1, 2, 3)$ that touches the xy- plane?

$r = 3$



$(x-1)^2 + (y-2)^2 + (z-3)^2 = 9$

Example 5: Find the equation of the sphere with radius 3 and center $(1, 4, 3)$. What is the intersection of this sphere with the three coordinate planes?

$(x-1)^2 + (y-4)^2 + (z-3)^2 = 9$

a.) Intersection with the xy-plane

in xy-plane $z = 0$

$(x-1)^2 + (y-4)^2 + 9 = 9$

$(x-1)^2 + (y-4)^2 = 0$ ← only true if $x=1, y=4$

$x = 1, \ y = 4$

Point: $(1, 4, 0)$

intersects in a point

b.) Intersection with the xz-plane

in xz-plane $y = 0$

$(x-1)^2 + 16 + (z-3)^2 = 9$

$(x-1)^2 + (z-3)^2 = -7$

never happens!

NO intersection

c.) Intersection with the yz-plane

x-coord of center is 1. smaller than radius

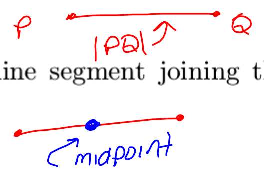
in yz-plane $x = 0$

$1 + (y-4)^2 + (z-3)^2 = 9$

$(y-4)^2 + (z-3)^2 = 8$

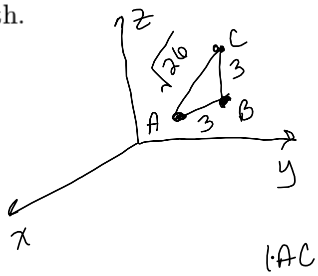
Circle of intersection

Distance formula in three dimensions: The distance between the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ is $|PQ| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$.



Midpoint of a line segment: The midpoint of the line segment joining the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ is $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$.

Example 6: Given the points $A(5, 5, 1)$, $B(3, 3, 2)$, and $C(1, 4, 4)$, determine whether ΔABC is isosceles, right, or both.



$|AB| = \sqrt{4+4+1} = 3$

$|BC| = \sqrt{4+1+4} = 3$

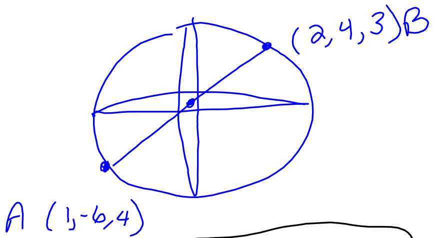
$|AB| = |BC|$ so ΔABC is isosceles

$|AC| = \sqrt{16+1+9} = \sqrt{26}$

$3^2 + 3^2 \overset{?}{=} (\sqrt{26})^2$ does not satisfy pyth. theorem. ΔABC is not right

Example 7: What is the equation of the sphere if one of its diameters has endpoints $(2, 4, 3)$ and $(1, -6, 4)$?

A B



center is midpoint

$C = \left(\frac{3}{2}, -1, \frac{7}{2}\right)$

$r = \frac{1}{2}|AB| = \frac{1}{2}\sqrt{1+100+1}$

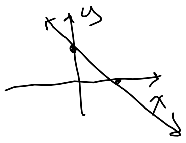
$r = \frac{\sqrt{102}}{2}$

$\left(x - \frac{3}{2}\right)^2 + (y+1)^2 + \left(z - \frac{7}{2}\right)^2 = \frac{102}{4}$

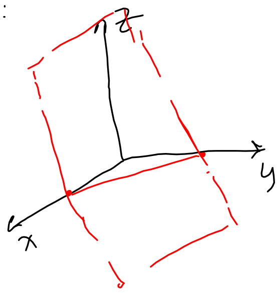
Example 8: Describe the following regions in the dimension indicated.

a.) $y = 2 - x$ in $\mathbb{R}^2$ and $\mathbb{R}^3$

$\mathbb{R}^2$: $y = 2 - x$ is a line



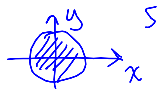
$\mathbb{R}^3$:



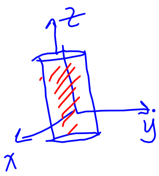
plane

b.) $x^2 + y^2 \leq 25$ in $\mathbb{R}^2$ and $\mathbb{R}^3$

$\mathbb{R}^2$: $x^2 + y^2 \leq 25$

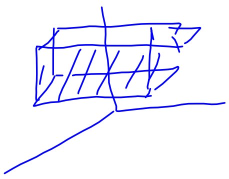


solid circle

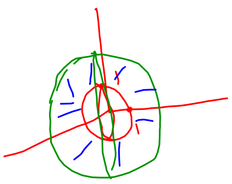


$\mathbb{R}^3$: solid cylinder

c.) $1 \leq z \leq 3$ in $\mathbb{R}^3$



d.) $1 < x^2 + y^2 + z^2 < 25$ in $\mathbb{R}^3$



between but not on the spheres $x^2 + y^2 + z^2 = 1$

$x^2 + y^2 + z^2 = 25$