Section 12.3: The Dot Product

So far, we have added two vectors and multiplied a vector by a scalar. Is it possible to multiply two vectors so that their product is a useful quantity? One such product is the dot product, which we will define in this section, and the other is called the cross, product which we will define in the next section.

Definition: If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, then $\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$.

Properties of the dot product. If **a**, **b** and **c** are vectors and c is a scalar, then

a.) $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$

b.) $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$

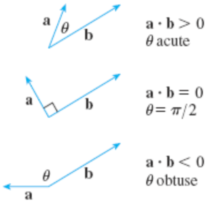
c.) $\mathbf{a} \cdot \mathbf{0} = \mathbf{0}$

d.) $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$

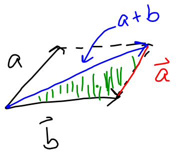
Example 1: Find the dot product between the vectors $\langle 1, 5, -2 \rangle$ and $\langle 0, 1, 4 \rangle$.

$$\langle 1,5,-2\rangle \cdot \langle 0,1,4\rangle = 1(0)+5(1)-2(4)$$
$$= 0+5-8$$
$$= \boxed{-3}$$

The dot product between two vectors can be given a geometric intrepretation in terms of the angle, θ, between **a** and **b**, $0 \le \theta \le \pi$.



$$\boxed{\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos\theta}$$



Example 2: Find $\vec{a} \cdot \vec{b}$ if it is known that $|\vec{a}| = 2$, $|\vec{b}| = 5$ and $\theta = 60°$.

$$a \cdot b = |a|\,|b| \cos\theta$$
$$= (2)(5)\cos(60^\circ)$$
$$= (2)(5)\left(\frac{1}{2}\right)$$
$$= \boxed{5}$$

Example 3: Find the angle between the vectors $\langle 1, 0, -2 \rangle$ and $\langle 2, -1, 3 \rangle$.

$$a \cdot b = |a|\,|b| \cos\theta$$

$$\boxed{\frac{a \cdot b}{|a|\,|b|} = \cos\theta}$$

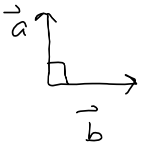
$$\cos\theta = \frac{\langle 1,0,-2\rangle \cdot \langle 2,-1,3\rangle}{\sqrt{1+4}\,\sqrt{4+1+9}}$$
$$= \frac{2-6}{\sqrt{5}\,\sqrt{14}}$$
$$\cos\theta = \frac{-4}{\sqrt{5}\,\sqrt{14}}$$
$$\theta = \arccos\left(\frac{-4}{\sqrt{70}}\right)$$
$$\theta \approx 119^\circ$$

Example 4: What is the value of $\vec{a} \cdot \vec{b}$ if it is known that $\vec{a}$ and $\vec{b}$ are perpendicular?

$$a\cdot b = |a||b|\cos\theta$$

$$= |a||b|\cos(90^\circ)$$

$$\boxed{a\cdot b = 0}$$



Example 5: What is the value of $\vec{a} \cdot \vec{b}$ if it is known that $\vec{a}$ and $\vec{b}$ are parallel?

$\theta = 0^\circ \rightarrow \cos(\theta) = 1$

$\theta = 180^\circ \rightarrow \cos(\theta) = -1$

$\cos\theta = \pm 1$

$$a\cdot b = |a||b|\overset{\pm 1}{\cancel{\cos\theta}}$$

$$\boxed{a\cdot b = \pm|a||b|}$$

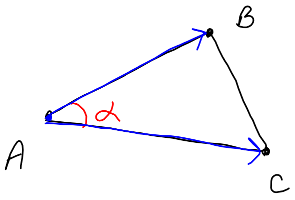
Example 6: For what value(s) of x are the vectors $\langle x, 1, 2\rangle$ and $\langle 3, 4, x\rangle$ perpendicular?

Solve $\langle x, 1, 2\rangle \cdot \langle 3, 4, x\rangle = 0$

$$3x + 4 + 2x = 0$$

$$5x = -4 \qquad \boxed{x = -\frac{4}{5}}$$

Example 7: The points $A(0,-1,6)$, $B(2,1,-3)$ and $C(5,4,2)$ form a triangle. Find the three angles of $\triangle ABC$.



$\vec{AB} = \langle 2, 2, -9\rangle$

$\vec{AC} = \langle 5, 5, -4\rangle$

$$\cos\alpha = \frac{\langle 2,2,-9\rangle \cdot \langle 5,5,-4\rangle}{\sqrt{4+4+81}\sqrt{25+25+16}}$$

$$= \frac{10+10+36}{\sqrt{89}\sqrt{66}}$$

$$\boxed{\alpha \approx 43^\circ}$$

For β:

$$\cos\beta = \frac{\vec{BA}\cdot\vec{BC}}{|\vec{BA}||\vec{BC}|}$$

$$\boxed{\beta \approx 58^\circ}$$

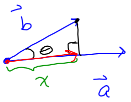
$$\gamma = 180^\circ - (\alpha + \beta)$$

$$\boxed{\gamma \approx 79^\circ}$$

Vector and Scalar Projections: Given $\vec{a} = \langle a_1, a_2, a_3\rangle$ and $\vec{b} = \langle b_1, b_2, b_3\rangle$, we want to project $\vec{a}$. Think of this as "the vector $\vec{b}$ in the direction of $\vec{a}$". Geometrically, to obtain the vector proje $\vec{b}$ onto $\vec{a}$, drop a perpendicular from the end of $\vec{b}$ onto $\vec{a}$.

(i) The **Scalar Projection** of $\vec{b}$ onto $\vec{a}$ (also called the component of $\vec{b}$ onto $\vec{a}$) is:

$$\boxed{\text{comp}_a b = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}}$$



vector projection of $\vec{b}$ onto $\vec{a}$

(ii) The **Vector Projection** of $\vec{b}$ onto $\vec{a}$ is:

$$\boxed{\text{proj}_a b = \left(\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}\right)\frac{\vec{a}}{|\vec{a}|}} = \left(\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2}\right)\vec{a}$$

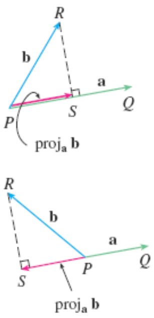
$$\cos\theta = \frac{x}{|b|}$$

$$x = |b|\cos\theta$$

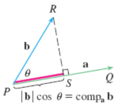
$$x = \cancel{|b|}\left(\frac{a\cdot b}{|a|\cancel{|b|}}\right)$$

$$\boxed{x = \frac{a\cdot b}{|a|}}$$

Vector projections



Scalar projection



Example 8: Find the vector and scalar projection of $\overset{\vec{b}}{\langle 1,-1,3\rangle}$ onto $\overset{\vec{a}}{\langle 0,2,1\rangle}$.

Scalar projection: $\text{comp}_a b = \dfrac{a\cdot b}{|a|} = \dfrac{\langle 1,-1,3\rangle\cdot\langle 0,2,1\rangle}{|\langle 0,2,1\rangle|}$

$$\boxed{\text{comp}_a b = \frac{1}{\sqrt{5}}}$$

vector projection: $\text{proj}_a b = \dfrac{1}{\sqrt{5}}\,\dfrac{\langle 0,2,1\rangle}{|\langle 0,2,1\rangle|}$

$$= \frac{1}{\sqrt{5}}\frac{\langle 0,2,1\rangle}{\sqrt{5}} = \left\langle 0, \frac{2}{5}, \frac{1}{5}\right\rangle$$

Example 9: If $\mathbf{a} = \langle 0,5,-2\rangle$, find a vector $\mathbf{b}$ so that $\boxed{\text{comp}_a b = 5.}$

Let $\vec{b} = \langle x, y, z\rangle$

$$\text{comp}_a b = \frac{a\cdot b}{|a|} = \frac{\langle 0,5,-2\rangle\cdot\langle x,y,z\rangle}{\sqrt{29}}$$

$$\frac{5y-2z}{\sqrt{29}} = 5$$

$$5y - 2z = 5\sqrt{29}$$

x can be anything, choose $y=0 \longrightarrow -2z = 5\sqrt{29}$

$$z = -\frac{5\sqrt{29}}{2}$$

one solution is $\langle x,y,z\rangle = \left\langle 0, 0, -\frac{5\sqrt{29}}{2}\right\rangle$