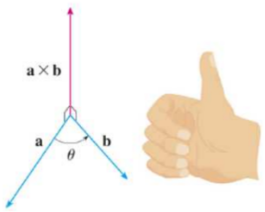
Section 12.4: The Cross Product

The **cross product** $\overrightarrow{a \times b}$ of two vectors $\vec{a}$ and $\vec{b}$ is a vector, unlike the dot product, which is a scalar. We will see later that $\overrightarrow{a \times b}$ is useful because it is perpendicular to both $\vec{a}$ and $\vec{b}$.



We can find $\overrightarrow{a \times b}$ in component form by using the fact that $\overrightarrow{a \times b}$ is perpendicular to both $\vec{a}$ and $\vec{b}$, along with the vector distributive laws (see page 668-669 in text for proof).

$$\overrightarrow{a \times b} = \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1 \rangle$$

In order to make this expression easier to remember, we will use the notation of determinates. A determinate of order 2 is defined by

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

A determinate of order 3 is defined by

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

Thus, if we make the first row **i**, **j**, and **k**, the second row **a**, and the third row **b**,

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \mathbf{i} \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \mathbf{j} \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \mathbf{k} \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

$$= \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1 \rangle$$

Example 1: Find the cross product of $\langle 1, 1, 3 \rangle$ and $\langle -2, -1, -5 \rangle$.

$$\begin{vmatrix} i & j & k \\ 1 & 1 & 3 \\ -2 & -1 & -5 \end{vmatrix}$$

$a \times b = -(b \times a)$

$\langle 1, 1, 3 \rangle \times \langle -2, -1, -5 \rangle = \langle -2, -1, 1 \rangle$

divide by magnitude

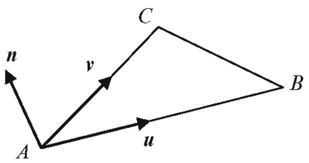
Example 2: Find a unit vector perpendicular to both $\langle 1, 2, 1\rangle$ and $\langle 0, 1, 3\rangle$.

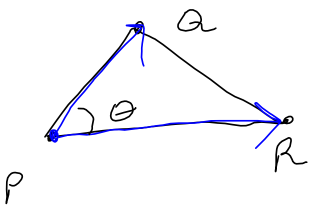
$\langle 1,2,1\rangle \times \langle 0,1,3\rangle$ is perpendicular to both vectors

$$\langle 1,2,1\rangle \times \langle 0,1,3\rangle = \begin{vmatrix} i & j & k \\ 1 & 2 & 1 \\ 0 & 1 & 3 \end{vmatrix} = \langle 5, -3, 1\rangle$$

$$\boxed{\vec{u} = \frac{\langle 5,-3,1\rangle}{\sqrt{25+9+1}} = \left\langle \frac{5}{\sqrt{35}}, \frac{-3}{\sqrt{35}}, \frac{1}{\sqrt{35}} \right\rangle}$$

Example 3: Find a vector perpendicular to the plane through the points $P(2, 3, 5)$, $Q(-1, 3, 4)$ and $R(3, 0, 6)$.





$\overrightarrow{PQ} = \langle -3, 0, -1\rangle$

$\overrightarrow{PR} = \langle 1, -3, 1\rangle$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} i & j & k \\ -3 & 0 & -1 \\ 1 & -3 & 1 \end{vmatrix} = \langle -3, 2, 9\rangle$$

Theorem: If θ is the angle between $\vec{a}$ and $\vec{b}$, then $|\overrightarrow{a \times b}| = |\vec{a}||\vec{b}|\sin\theta$.

$\boxed{a \cdot b = |a||b|\cos\theta}$ to find θ

Corollary: Two non zero vectors $\vec{a}$ and $\vec{b}$ are parallel if and only if $|\overrightarrow{a \times b}| = 0$.

why? $\vec{a}$ parallel to $\vec{b}$ means $\sin\theta = 0$

$|a \times b| = |a||b|(0)$

Example 4: Find the angle between the vectors $\vec{a} = 2\vec{i} + 6\vec{j} - 4\vec{k}$ and $\vec{b} = -3\vec{i} - 9\vec{j} + 6\vec{k}$.

$a \cdot b = |a||b|\cos\theta$

$$\cos\theta = \frac{\langle 2,6,-4\rangle \cdot \langle -3,-9,6\rangle}{\sqrt{4+36+16}\,\sqrt{9+81+36}} = \frac{-84}{\sqrt{56}\,\sqrt{126}} = -1$$

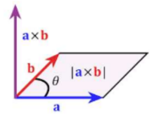
vectors are parallel

$\theta = 180°$

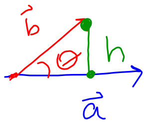
or show $a \times b = \langle 0, 0, 0\rangle$

Note: we can always find θ using the formula $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos\theta$.

Theorem: The length of the cross product $\vec{a} \times \vec{b}$ is equal to the area of the parallelogram determined by **a** and **b**.



Area parallelogram is $A = (\text{base})(\text{height})$



$A = |a|\,h$

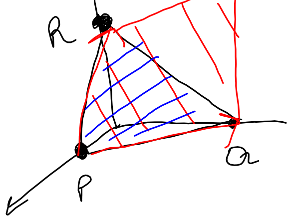
$\sin\theta = \dfrac{h}{|b|}$ $\quad h = |b|\sin\theta$

$A = |a||b|\sin\theta$

$A = |a \times b|$

formula on previous page

Example 5: Find the area of the triangle determined by the points $P(1,0,0)$, $Q(0,2,0)$ and $R(0,0,3)$.



$A_\Delta = \frac{1}{2} A_{\text{parallelogram}}$

$= \frac{1}{2}\left| \vec{PR} \times \vec{PQ} \right|$

$PR = \langle -1, 0, 3 \rangle$

$PQ = \langle -1, 2, 0 \rangle$

$PR \times PQ = \begin{vmatrix} i & j & k \\ -1 & 0 & 3 \\ -1 & 2 & 0 \end{vmatrix} = \langle -6, -3, -2 \rangle$

$A_\Delta = \frac{1}{2}\left| \langle -6, -3, -2 \rangle \right|$

$= \frac{1}{2}\sqrt{36+9+4} = \frac{1}{2}\sqrt{49} = \boxed{\frac{7}{2}}$

Example 6: Determine whether each expression is meaningful or meaningless (circle one). If so, state whether the expression is a vector or a scalar.

a.) $\mathbf{a} \cdot \mathbf{b}$	meaningful (vector or scalar)	meaningless
b.) $\mathbf{a} \times \mathbf{b}$	meaningful (vector or scalar)	meaningless
c.) $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$	meaningful (vector or scalar)	meaningless
d.) $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$	meaningful (vector or scalar)	meaningless
e.) $(\mathbf{a} \cdot \mathbf{b}) \times (\mathbf{c} \cdot \mathbf{d})$ cannot cross numbers	meaningful (vector or scalar)	meaningless
f.) $(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d})$	meaningful (vector or scalar)	meaningless
g.) $\mathbf{a} \times (\mathbf{b} \cdot \mathbf{c})$	meaningful (vector or scalar)	meaningless
h.) $\lvert\mathbf{a}\rvert(\mathbf{b} \times \mathbf{c})$	meaningful (vector or scalar)	meaningless