Section 12.5: Equations of Lines and Planes

I. **Lines:** A line in the xy-plane is determined when point on the line and the direction of the line (its slope or angle of inclination) are given. The equation of the line can then be written in point-slope form: $y - y_1 = m(x - x_1)$.

Likewise, a line L in three-dimensional space is determined when we know a point $P_0(x_0, y_0, x_0)$ on L and the *direction* of L. The direction of a line in space is *any* vector parallel to the line.

Definition Let $r_0 = (x_0, y_0, x_0)$ be any point on L and let $\mathbf{v} = \langle a, b, c\rangle$ be any vector parallel to the line (we call this the **direction vector** of the line). The **vector equation of the line** is

$$\mathbf{r} = \mathbf{r_0} + t\mathbf{v}$$

As the parameter t varies, the above equation traces out the line.

(i) We call $\mathbf{r} = \mathbf{r_0} + t\mathbf{v} = \langle x_0, y_0, z_0\rangle + t\langle a, b, c\rangle = \langle x_0 + ta, y_0 + tb, z_0 + tc\rangle$ a **vector equation** of the line.

(ii) We call

$$x = x_0 + ta, y = y_0 + tb, z = z_0 + tc$$

parametric equations of the line. NOTE: If parametric equations of a line are given as $x = x_0 + ta$, $y = y_0 + tb$, $z = z_0 + tc$, then then a direction vector of the line is the vector of coefficients in front of t, in this case $\langle a, b, c\rangle$.

(iii) By solving the three parametric equations for t, we call

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

symmetric equations of the line, provided a, b and $c \neq 0$.

If, for example, $a = 0$, then $x = x_0$ in which case the symmetric equations would be $x = x_0$, $\dfrac{y - y_0}{b} = \dfrac{z - z_0}{c}$.

Example 1: Find a vector equation and a set of parametric equations for the line passing through the point $(1, 2, -3)$ and parallel to the vector $\langle 1, 5, 6\rangle$. Where does this line intersect the xy-plane?

Example 2: Find an equation of the line (in any form) passing through the point $(2, 1, 4)$ and parallel to the line $x = 1 + 4t$, $y = 3 - 6t$, $z = 4 + 5t$.

Example 3: Find symmetric equations for the line passing through the points $(1, 3, -4)$ and $(0, 1, 5)$.

Definition: Two lines are **skew** if the lines do not intersect and are not parallel.

Example 4: Consider $x = 1 + t$, $y = -2 + 3t$, $z = 4 - t$ and $x = 2v$, $y = 3 + v$, $z = -3 + 4v$. Determine whether the lines are parallel, intersecting, or skew. If the lines intersect, find the point of intersection.

Example 5: Consider the lines: $\dfrac{x-1}{2} = y = \dfrac{z-1}{4}$ and $x = \dfrac{y+2}{2} = \dfrac{z+2}{3}$.

Determine whether these are lines parallel, intersecting, or skew. If the lines intersect, find the point of intersection.

II. **Planes:** A plane in space is determined by a point on the plane $P = (x_0, y_0, z_0)$ and any vector $\mathbf{n} = \langle a, b, c\rangle$ perpendicular to the plane. Let $\mathbf{n} = \langle a, b, c\rangle$ be any vector perpendicular to the plane (**called a normal vector**), let $P_0(x_0, y_0, z_0)$ be any point in the plane and $P(x, y, z)$ be any arbitrary point in the plane. Let $\mathbf{r_0}$ and $\mathbf{r}$ be the corresponding position vectors of the points P_0 and P, respectively.

Then the vector $\mathbf{r} - \mathbf{r_0}$ lies in the plane, and hence $\mathbf{r} - \mathbf{r_0}$ must be perpendicular to $\mathbf{n}$.

Thus $\mathbf{n}\cdot(\mathbf{r} - \mathbf{r_0}) = 0 \Rightarrow \langle a, b, c\rangle \cdot \langle x - x_0, y - y_0, z - z_0\rangle = 0$, which yields $a(x-x_0)+b(y-y_0)+c(z-z_0) = 0$.

Hence the **equation of the plane** with normal vector $\mathbf{n} = \langle a, b, c\rangle$ and passing through the point is

$$\mathbf{n}\cdot(\mathbf{r} - \mathbf{r_0}) = 0$$

Example 6: Find an equation of the plane passing through the point $(1, -1, 2)$ and perpendicular to $\langle -1, 3, 0\rangle$.

Example 7: Find an equation of the plane passing through the point $(2, 0, 3)$ and parallel to $2x+3y-4z = 11$.

Example 8: Find an equation of the plane passing through the points $(0, 1, 0)$, $(1, 2, -1)$ and $(0, 0, 3)$.

Example 9: Find an equation of the plane passing through the point $(-1, -3, 2)$ that contains the line $x = -1 - 2t$, $y = 4t$, $z = 2 + t$.

Example 10: Find an equation of the plane passing through the point $(-3, 1, 4)$ perpendicular to the line $x = 2 - 3t$, $y = 3 - t$, $z = t$.

Example 11: Where does the line $x = 1 + t$, $y = 2t$, $z = 3t$ intersect the plane $x + y + z = 1$?

Angle between two planes: Two planes are parallel (perpendicular) if their corresponding normal vectors are parallel (perpendicular). Moreover, the angle beteween two planes is defined to be the angle between their normal vectors.

Example 12: Find the angle between the planes $x + z = 1$ and $y + z = 1$.

The intersection of two planes: If two planes are not parallel, then they intersect in a line.

Example 13: Find the line of intersection of the planes

$z = x + y$ and $2x - 5y - z = 1$.

Distance from a point to a plane: The distance from the point (x_1, y_1, z_1) to the plane $ax+by+cz+d = 0$ is

$$D = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

Example 14: Find the distance from the point $(3, -2, 7)$ to the plane $4x - 6y + z = 5$.