Section 12.5: Equations of Lines and Planes

I. **Lines:** A line in the xy-plane is determined when point on the line and the direction of the line (its slope or angle of inclination) are given. The equation of the line can then be written in point-slope form: $\boxed{y - y_1 = m(x - x_1).}$

$y = mx + b \quad m = \text{slope}, \quad b = y\text{-intercept}$

Likewise, a line L in three-dimensional space is determined when we know a point $P_0(x_0, y_0, z_0)$ on L and the *direction* of L. The direction of a line in space is *any* vector parallel to the line.

Definition Let $r_0 = (x_0, y_0, z_0)$ be any point on L and let $\mathbf{v} = \langle a, b, c\rangle$ be any vector parallel to the line (we call this the **direction vector** of the line). The **vector equation of the line** is

$$\boxed{\mathbf{r} = \mathbf{r_0} + t\mathbf{v}}$$

$r_0 = (x_0, y_0, z_0)$ any point on line.

As the parameter t varies, the above equation traces out the line.

$\vec{v} = \langle a, b, c\rangle$ that is parallel to the line.

$r(t) = \vec{r}_0 + t\vec{v}$

$\vec{r}_0$: point vector

$\vec{v}$: slope vector

$= \langle x_0, y_0, z_0\rangle + t\langle a, b, c\rangle \leftarrow$ vector equation of the line.

$= \langle x_0 + ta, y_0 + tb, z_0 + tc\rangle \leftarrow$

(i) We call $\mathbf{r} = \mathbf{r_0} + t\mathbf{v} = \langle x_0, y_0, z_0\rangle + t\,\langle a, b, c\rangle = \langle x_0 + ta, y_0 + tb, z_0 + tc\rangle$ a **vector equation** of the line.

(ii) We call

$$\boxed{x = x_0 + ta, y = y_0 + tb, z = z_0 + tc} \leftarrow \text{parametric equation of line}$$

parametric equations of the line. NOTE: If parametric equations of a line are given as $x = x_0 + ta$, $y = y_0 + tb$, $z = z_0 + tc$, then then a direction vector of the line is the vector of coefficients in front of t, in this case $\langle a, b, c\rangle$.

$x = x_0 + ta$, $\quad y = y_0 + tb$

$t = \dfrac{x - x_0}{a}$ $\qquad t = \dfrac{y - y_0}{b}$

(iii) By solving the three parametric equations for t, we call

$$\boxed{\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}}$$

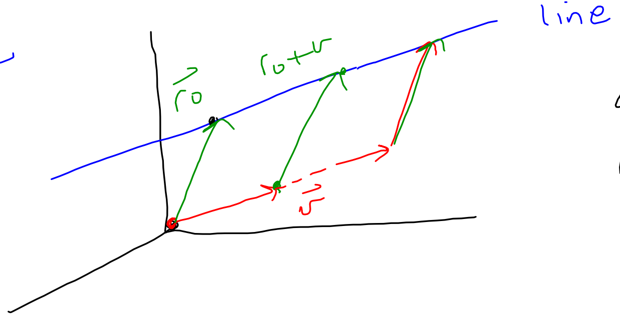
$z = z_0 + tc$

$t = \dfrac{z - z_0}{c}$

symmetric equations of the line, provided a, b and $c \neq 0$.

If, for example, $a = 0$, then $x = x_0$ in which case the symmetric equations would be $x = x_0$, $\dfrac{y - y_0}{b} = \dfrac{z - z_0}{c}$.

picture



as t varies, $\vec{r}_0 + t\vec{v}$ lies on line

Example 1: Find a vector equation and a set of parametric equations for the line passing through the point $(1, 2, -3)$ and parallel to the vector $\langle 1, 5, 6\rangle$. Where does this line intersect the xy-plane?

equation of a line is $r(t) = \vec{r_0} + t\vec{v}$

$\vec{r_0} = \langle 1, 2, -3\rangle$

$\vec{v} = \langle 1, 5, 6\rangle$

$\vec{r_0}$ = point vector

$\vec{v}$ = slope vector = vector parallel to line

$r(t) = \langle 1, 2, -3\rangle + t\langle 1, 5, 6\rangle$ ← vector equation of line

$= \langle 1+t, 2+5t, -3+6t\rangle$

Parametric equations: $x = 1+t$, $y = 2+5t$, $z = -3+6t$

intersects the xy plane when $z = 0$

$-3 + 6t = 0$ $\quad t = \frac{1}{2}$

$x = 1 + \frac{1}{2} = \frac{3}{2}$

$y = 2 + \frac{5}{2} = \frac{9}{2}$

point where it intersects the xy plane is $\left(\frac{3}{2}, \frac{9}{2}, 0\right)$

Example 2: Find an equation of the line (in any form) passing through the point $(2, 1, 4)$ and $\boxed{\text{parallel}}$ to the line $\boxed{x = 1 + 4t,\ y = 3 - 6t,\ z = 4 + 5t.}$

note: $r_0 + tv$

slope of the line is the vector coefficients of t

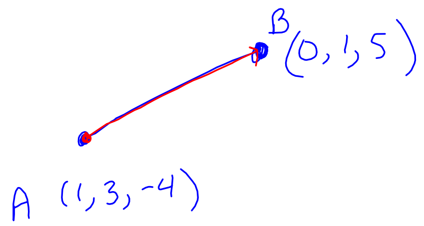
$v = \langle 4, -6, 5\rangle$

parallel slope vectors

the line passing thru $(2,1,4)$ parallel to the given line is

$$\boxed{\vec{r_0} + t\vec{v} = \langle 2, 1, 4\rangle + t\langle 4, -6, 5\rangle}$$

Example 3: Find symmetric equations for the line passing through the points $(1, 3, -4)$ and $(0, 1, 5)$.



r_0 = either point

$\vec{r_0} = \langle 1, 3, -4\rangle$ (or $\langle 0,1,5\rangle$)

$\vec{v} = \vec{AB} = \langle -1, -2, 9\rangle$ (or $\vec{BA}$)

$\vec{r_0} + t\vec{v} = \langle 1, 3, -4\rangle + t\langle -1, -2, 9\rangle$

$x = 1 - t \rightarrow t = 1 - x$

$y = 3 - 2t \rightarrow t = \frac{y-3}{-2}$

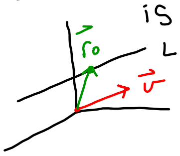
$z = -4 + 9t \rightarrow t = \frac{z+4}{9}$

symmetric equation of line

$1 - x = \frac{y-3}{-2} = \frac{z+4}{9}$

Recall The equation of a line in space is $r(t) = \vec{r}_0 + t\vec{v}$, $\vec{r}_0$ = "point vector"

$\vec{v}$ = any vector parallel to the line "slope vector"



Definition: Two lines are skew if the lines do not intersect and are not parallel.

Example 4: Consider $x = 1+t$, $y = -2+3t$, $z = 4-t$ and $x = 2v$, $y = 3+v$, $z = -3+4v$. Determine whether the lines are parallel, intersecting, or skew. If the lines intersect, find the point of intersection.

L_1: $\boxed{x = 1+t,\ y = -2+3t,\ z = 4-t}$ $\quad v_1 = \langle 1, 3, -1\rangle$

L_2: $x = 2v$, $y = 3+v$, $z = -3+4v$ $\quad v_2 = \langle 2, 1, 4\rangle$

Two lines are parallel if if slope vectors are parallel

v_1 not parallel to v_2. Lines are not parallel.

Do they intersect?

$\boxed{1+t = 2v} \rightarrow \boxed{t = 2v-1}$

$-2+3t = 3+v$

$4-t = -3+4v$

$\rightarrow 4-2v+1 = -3+4v$

$8 = 6v$

$\boxed{v = \frac{4}{3}}$

$\boxed{t = \frac{8}{3} - 1 = \frac{5}{3}}$

$\boxed{t = \frac{5}{3},\ v = \frac{4}{3}}$

$-2 + 3\left(\frac{5}{3}\right) = 3 + \frac{4}{3}$

$3 = \frac{13}{3}$ FALSE $\quad$ $\boxed{\text{lines are skew}}$

Example 5: Consider the lines: $\frac{x-1}{2} = y = \frac{z-1}{4}$ and $x = \frac{y+2}{2} = \frac{z+2}{3}$.

Determine whether these are lines parallel, intersecting, or skew. If the lines intersect, find the point of intersection.

L_1: $\frac{x-1}{2} = y = \frac{z-1}{4}$ $\quad t=0$

$\frac{x-1}{2} = t \rightarrow x = 2t+1$

$y = t \rightarrow y = t$

$\frac{z-1}{4} = t \rightarrow z = 4t+1$

L_2: $x = \frac{y+2}{2} = \frac{z+2}{3}$ $\quad v=1$

$x = v$ $\quad$ $x = v$

$\frac{y+2}{2} = v$ $\quad$ $y = 2v-2$

$\frac{z+2}{3} = v$ $\quad$ $z = 3v-2$

$v_1 = \langle 2, 1, 4\rangle$ $\quad$ $v_2 = \langle 1, 2, 3\rangle$ $\quad$ not parallel

do they intersect?

$\boxed{2t+1 = v}$

$t = 2v-2 \rightarrow t = 2(2t+1)-2$

$4t+1 = 3v-2$ $\quad$ $t = 4t+2-2$

$t = 4t$

$\boxed{t=0 \quad v=1}$

$4(0)+1 = 3(1)-2$

$1 = 1$ ✓

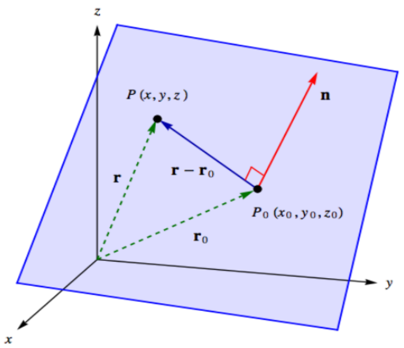
Find point of intersection:

plug $t=0$ into L_1

or $v=1$ into L_2

$\boxed{\text{Both yield } (1,0,1)}$

II. **Planes:** A plane in space is determined by a point on the plane $P = (x_0, y_0, z_0)$ and any vector $\mathbf{n} = \langle a, b, c\rangle$ perpendicular to the plane. Let $\mathbf{n} = \langle a, b, c\rangle$ be any vector perpendicular to the plane (**called a normal vector**), let $P_0(x_0, y_0, z_0)$ be any point in the plane and $P(x, y, z)$ be any arbitrary point in the plane. Let $\mathbf{r_0}$ and $\mathbf{r}$ be the corresponding position vectors of the points P_0 and P, respectively.



$\vec{n}\cdot(\overrightarrow{r-r_0})=0$

Then the vector $\mathbf{r} - \mathbf{r_0}$ lies in the plane, and hence $\mathbf{r} - \mathbf{r_0}$ must be perpendicular to $\mathbf{n}$.

Thus $\mathbf{n}\cdot(\mathbf{r} - \mathbf{r_0}) = 0 \Rightarrow \langle a, b, c\rangle \cdot \langle x - x_0, y - y_0, z - z_0\rangle = 0$, which yields $a(x-x_0)+b(y-y_0)+c(z-z_0) = 0$.

Hence the **equation of the plane** with normal vector $\mathbf{n} = \langle a, b, c\rangle$ and passing through the point is

$$\mathbf{n}\cdot(\mathbf{r} - \mathbf{r_0}) = 0$$

Example 6: Find an equation of the plane passing through the point $(1, -1, 2)$ and perpendicular to $\langle -1, 3, 0\rangle$.

A plane requires
1. point $r_0(x_0, y_0, z_0)$ in plane
2. point $r(x, y, z)$ ← arbitrary
3. vector perpendicular to the plane $n = \langle a, b, c\rangle$

$\vec{n}\cdot(\overrightarrow{r-r_0})=0$

$\vec{r_0} = \langle 1, -1, 2\rangle$

$\vec{r} = \langle x, y, z\rangle$

$n = \langle -1, 3, 0\rangle$

$\langle -1,3,0\rangle\cdot\langle x-1, y+1, z-2\rangle = 0$

$-1(x-1)+3(y+1)+0(z-2)=0$

$-x+1+3y+3=0$

$\boxed{-x+3y=-4}$

Example 7: Find an equation of the plane passing through the point $(2,0,3)$ and parallel to $\underbrace{2x+3y-4z=11}$.

$\vec{n} = \langle 2,3,-4 \rangle$

Two planes are parallel if their normal vectors are parallel to each other.

$\vec{r}_0 = \langle 2,0,3 \rangle$

$\vec{r} = \langle x,y,z \rangle$

$\vec{n} = \langle 2,3,-4 \rangle$

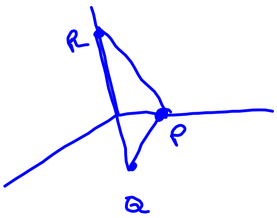
$\vec{n}\cdot(\vec{r}-\vec{r}_0)=0 \rightarrow \langle 2,3,-4 \rangle \cdot \langle x-2, y-0, z-3 \rangle = 0$

$2x-4+3y-4z+12=0$

$\boxed{2x+3y-4z=-8}$

Example 8: Find an equation of the plane passing through the points $(0,1,0)$, $(1,2,-1)$ and $(0,0,3)$.

P Q R



$\vec{r}_0 = \langle 0,1,0 \rangle$

$\vec{n} = \vec{PQ} \times \vec{PR}$

$\vec{n} = \langle 1,1,-1 \rangle \times \langle 0,-1,3 \rangle$

$= \begin{vmatrix} i & j & k \\ 1 & 1 & -1 \\ 0 & -1 & 3 \end{vmatrix}$

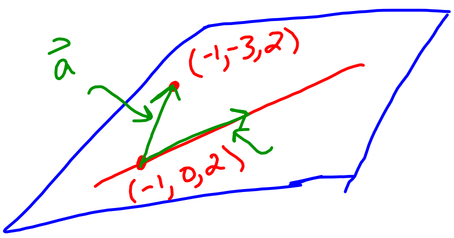
$\boxed{\vec{n} = \langle 2,-3,-1 \rangle}$

$\vec{n}\cdot(\vec{r}-\vec{r}_0)=0 \rightarrow \langle 2,-3,-1 \rangle \cdot \langle x, y-1, z \rangle = 0$

$2x-3y+3-z=0$ $\boxed{2x-3y-z=-3}$

Example 9: Find an equation of the plane passing through the point $(-1,-3,2)$ that contains the line $\underbrace{x=-1-2t,\ y=4t,\ z=2+t}$.

$\vec{r}_0 = \langle -1,-3,2 \rangle$



$\vec{a} = \langle 0,-3,0 \rangle$

$v = \langle -2,4,1 \rangle$

direction vector of line.

$\vec{n} = \langle 0,-3,0 \rangle \times \langle -2,4,1 \rangle$

$= \begin{vmatrix} i & j & k \\ 0 & -3 & 0 \\ -2 & 4 & 1 \end{vmatrix}$

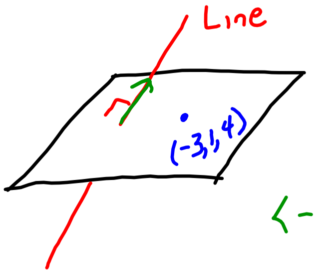
$\vec{n} = \langle -3,0,-6 \rangle$

$\langle -3,0,-6 \rangle \cdot \langle x+1, y+3, z-2 \rangle = 0$

$-3x-3-6z+12=0$

$\boxed{-3x-6z=-9}$

Example 10: Find an equation of the plane passing through the point $(-3, 1, 4)$ perpendicular to the line $x = 2 - 3t$, $y = 3 - t$, $z = t$.



$\vec{n} = \langle -3, -1, 1 \rangle$

$\vec{r}_0 = \langle -3, 1, 4 \rangle$

$\langle -3, -1, 1 \rangle \cdot \langle x+3, y-1, z-4 \rangle = 0$

$-3x - 9 - y + 1 + z - 4 = 0$

Example 11: Where does the line $x = 1 + t$, $y = 2t$, $z = 3t$ intersect the plane $x + y + z = 1$?

$1 + t + 2t + 3t = 1$

$6t = 0 \rightarrow t = 0$, point $(1, 0, 0)$

Angle between two planes: Two planes are parallel (perpendicular) if their corresponding normal vectors are parallel (perpendicular). Moreover, the angle beteween two planes is defined to be the angle between their normal vectors.

Example 12: Find the angle between the planes $x + z = 1$ and $y + z = 1$.

$n_1 = \langle 1, 0, 1 \rangle$

$n_2 = \langle 0, 1, 1 \rangle$

$\cos\theta = \dfrac{\langle 1,0,1 \rangle \cdot \langle 0,1,1 \rangle}{\sqrt{2}\sqrt{2}}$

$\cos\theta = \dfrac{1}{2}$ $\quad \theta = 60^\circ$

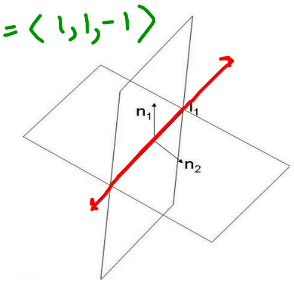
The intersection of two planes: If two planes are not parallel, then they intersect in a line.

Example 13: Find the line of intersection of the planes

$z = x + y$ and $2x - 5y - z = 1$.

$x + y - z = 0$ $\quad \langle 2, -5, -1 \rangle$

$n_1 = \langle 1, 1, -1 \rangle$



Recall: The equation of a line is

$r(t) = \vec{r}_0 + t\vec{v}$

r_0 lies in both planes

$z = x + y$

$2x - 5y - z = 1$

choose $\boxed{y = 0}$ in both planes

$z = x$

$2x - z = 1$

$2x - x = 1$

$x = 1$ $\quad z = 1$

point on line of intersection is $\boxed{r_0\ (1, 0, 1)}$

$v = \overrightarrow{n_1 \times n_2} = \langle 1, 1, -1 \rangle \times \langle 2, -5, -1 \rangle$

$$= \begin{vmatrix} i & j & k \\ 1 & 1 & -1 \\ 2 & -5 & -1 \end{vmatrix}$$

$$\boxed{\vec{v} = \langle -6, -1, -7 \rangle}$$

$$\boxed{\begin{aligned} r(t) &= \vec{r}_0 + t\vec{v} \\ &= \langle 1, 0, 1 \rangle + t\langle -6, -1, -7 \rangle \end{aligned}}$$

Distance from a point to a plane: The distance from the point (x_1, y_1, z_1) to the plane $ax + by + cz + d = 0$ is

$$D = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

Example 14: Find the distance from the point $(3, -2, 7)$ to the plane $4x - 6y + z = 5$.

$4x - 6y + z - 5 = 0$

$a = 4$	$x_1 = 3$
$b = -6$	$y_1 = -2$
$c = 1$	$z_1 = 7$
$d = -5$	

$$D = \left| \frac{4(3) + -6(-2) + 1(7) - 5}{\sqrt{16 + 36 + 1}} \right|$$

$$= 26/\sqrt{53}$$