13.1 (Vector Functions/Space Curves) and 13.2 (Derivatives/Integrals of Vector Functions

Definition: Let $\overrightarrow{r}$ be a vector function whose range is a set of three-dimensional vectors. This means that for every number t in the domain of $\overrightarrow{r}$, there is a unique vector $\overrightarrow{r(t)} = \langle f(t), g(t), h(t)\rangle$. We use t as the independent variable because it represents time in most applications of vector functions.

Example 1: Find $\overrightarrow{r(e)}$ where $\overrightarrow{r(t)} = \langle \cos t, \sin t, \ln t\rangle$.

Definition: The limit of a vector function is defined by taking the limit of the component functions, that is if $\overrightarrow{r(t)} = \langle f(t), g(t), h(t)\rangle$, then $\lim\limits_{t\to a} \overrightarrow{r(t)} = \left\langle \lim\limits_{t\to a} f(t), \lim\limits_{t\to a} g(t), \lim\limits_{t\to a} h(t)\right\rangle$.

Example 2: Find $\lim\limits_{t\to\infty} \left\langle \dfrac{1-e^{-3t}}{4t}, \arctan t, \ln\dfrac{3+2t}{5t-3}\right\rangle$.

Definition: If $\overrightarrow{r(t)} = \langle f(t), g(t), h(t)\rangle$, where f, g and h are differentiable functions, then

$\overrightarrow{r'(t)} = \langle f'(t), g'(t), h'(t)\rangle$.

Theorem: Suppose **u** and **v** are differentiable functions, c is a scalar, and f is a real valued function. Then

1. $\dfrac{d}{dt}\left(\mathbf{u}(\mathbf{t}) + \mathbf{v}(\mathbf{t})\right) = \mathbf{u}'(t) + \mathbf{v}'(t)$
2. $\dfrac{d}{dt}c\mathbf{u}(t) = c\mathbf{u}'(t)$
3. $\dfrac{d}{dt}\left(\mathbf{u}(\mathbf{t}) \cdot \mathbf{v}(\mathbf{t})\right) = \mathbf{u}'(t)\cdot\mathbf{v}(t) + \mathbf{u}(t)\cdot\mathbf{v}'(t)$
4. $\dfrac{d}{dt}\left(\mathbf{u}(\mathbf{t}) \times \mathbf{v}(\mathbf{t})\right) = \mathbf{u}'(t)\times\mathbf{v}(t) + \mathbf{u}(t)\times\mathbf{v}'(t)$
5. $\dfrac{d}{dt}\mathbf{u}(f(t)) = f'(t)\mathbf{u}'(t)$

Example 3: Find the domain of both $\overrightarrow{r(t)}$ and $\overrightarrow{r'(t)}$ if $\overrightarrow{r(t)} = \left\langle t\cos t, \sqrt{\ln t}, \dfrac{t}{t-2}\right\rangle$.

Example 4: Find the **Unit Tangent Vector** (tangent vector of length one) to the curve $\overrightarrow{r(t)} = \langle e^{2t}, t, te^t \rangle$ at $t = 0$.

Example 5: Find parametric equations for the tangent line to the curve $x = 2\cos t$, $y = 2\sin t$, $z = 4\cos(2t)$ at the point $(\sqrt{3}, 1, 2)$.

Definition: If $\overrightarrow{r(t)} = \langle f(t), g(t), h(t) \rangle$, where f, g and h are integratible functions, then

(i) $\displaystyle\int \overrightarrow{r(t)}\, dt = \left\langle \int f(t)\, dt, \int g(t)\, dt, \int h(t)\, dt \right\rangle$

(ii) $\displaystyle\int_a^b \overrightarrow{r(t)}\, dt = \left\langle \int_a^b f(t)\, dt, \int_a^b g(t)\, dt, \int_a^b h(t)\, dt \right\rangle$

Example 6: Find $\displaystyle\int \left(te^{3t}\mathbf{i} + t\ln t\mathbf{j} + t\cos(t^2)\mathbf{k} \right)\, dt$

Example 7: Find $\overrightarrow{r(t)}$ if $\overrightarrow{r'(t)} = \left\langle \sin t, e^{-2t}, \dfrac{1}{t^2+1} \right\rangle$ and $\overrightarrow{r(0)} = \langle 2, 1, -2 \rangle$.

Example 8: Evaluate $\int_0^1 \left(t^2\mathbf{i} + t\mathbf{j} + \frac{1}{t+1}\mathbf{k} \right) dt$

Example 9: At what points does the curve $\overrightarrow{r(t)} = t\mathbf{i} + (2t - t^2)\mathbf{k}$ intersect the paraboloid $z = x^2 + y^2$?

Definition: The set C of all points $(f(t), g(t), h(t))$, where t varies over its domain, is called a **space curve.**

Example 10: Match the parametric equations with the graphs (labeled I-VI)

a. $x = t\cos t,\ y = t,\ z = t\sin t,\ t \geq 0$

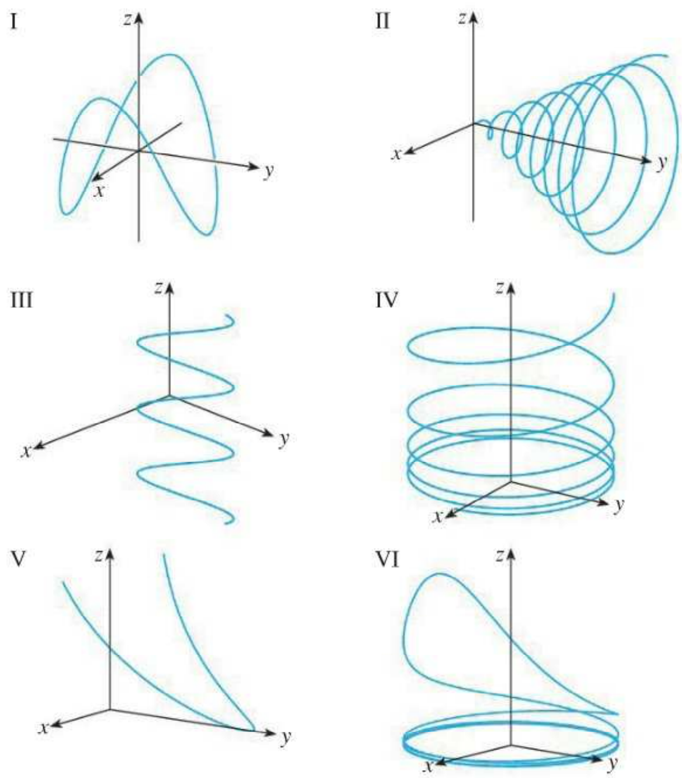
b. $x = \cos t,\ y = \sin t,\ z = \dfrac{1}{1+t^2}$

c. $x = t,\ y = \dfrac{1}{1+t^2},\ z = t^2$

d. $x = \cos t,\ y = \sin t,\ z = \cos(2t)$

e. $x = \cos 8t,\ y = \sin 8t,\ z = e^{0.8t}$

f. $x = \cos^2 t,\ y = \sin^2 t,\ z = t$



Example 10: At what point do the curves $\overrightarrow{r_1(t)} = \langle t, 1-t, 3+t^2 \rangle$ and $\overrightarrow{r_2(s)} = \langle 3-s, s-2, s^2 \rangle$ intersect? Find the angle of intersection correct to the nearest degree.