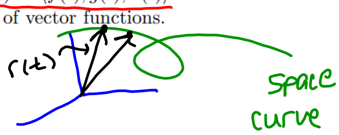
13.1 (Vector Functions/Space Curves) and 13.2 (Derivatives/Integrals of Vector Functions

Definition: Let $\vec{r}$ be a vector function whose range is a set of three-dimensional vectors. This means that for every number t in the domain of $\vec{r}$, there is a unique vector $\overrightarrow{r(t)} = \langle f(t), g(t), h(t)\rangle$. We use t as the independent variable because it represents time in most applications of vector functions.

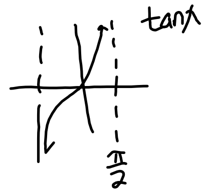
Example 1: Find $\overrightarrow{r(e)}$ where $\overrightarrow{r(t)} = \langle \cos t, \sin t, \ln t\rangle$.

$\overrightarrow{r(e)} = \langle \cos e, \sin e, \cancel{\ln e}\ 1 \rangle$



Definition: The limit of a vector function is defined by taking the limit of the component functions, that is if $\overrightarrow{r(t)} = \langle f(t), g(t), h(t)\rangle$, then $\lim_{t\to a} \overrightarrow{r(t)} = \left\langle \lim_{t\to a} f(t), \lim_{t\to a} g(t), \lim_{t\to a} h(t)\right\rangle$.

Example 2: Find $\lim_{t\to\infty} \left\langle \frac{1-e^{-3t}}{4t}, \arctan t, \ln\frac{3+2t}{5t-3}\right\rangle$.



① $\lim_{t\to\infty} \frac{1-e^{-3t}}{4t} = \lim_{t\to\infty} \frac{1-\frac{1}{e^{3t}}}{4t} = \frac{1}{\infty} = 0$

② $\lim_{t\to\infty} \arctan t = \frac{\pi}{2}$

③ $\lim_{t\to\infty} \ln\left(\frac{3+2t}{5t-3}\right) = \ln\left(\frac{2}{5}\right)$

$\boxed{\text{Answer: } \langle 0, \frac{\pi}{2}, \ln\frac{2}{5}\rangle}$

Definition: If $\overrightarrow{r(t)} = \langle f(t), g(t), h(t)\rangle$, where f, g and h are differentiable functions, then $\overrightarrow{r'(t)} = \langle f'(t), g'(t), h'(t)\rangle$.

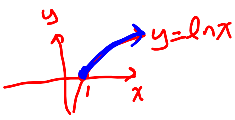
Theorem: Suppose **u** and **v** are differentiable functions, c is a scalar, and f is a real valued function. Then

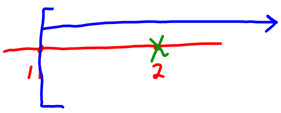
1. $\frac{d}{dt}(\mathbf{u(t)} + \mathbf{v(t)}) = \mathbf{u}'(t) + \mathbf{v}'(t)$
2. $\frac{d}{dt}c\mathbf{u}(t) = c\mathbf{u}'(t)$
3. $\frac{d}{dt}(\mathbf{u(t)} \cdot \mathbf{v(t)}) = \mathbf{u}'(t)\cdot\mathbf{v}(t) + \mathbf{u}(t)\cdot\mathbf{v}'(t)$
4. $\frac{d}{dt}(\mathbf{u(t)} \times \mathbf{v(t)}) = \mathbf{u}'(t)\times\mathbf{v}(t) + \mathbf{u}(t)\times\mathbf{v}'(t)$
5. $\frac{d}{dt}\mathbf{u}(f(t)) = f'(t)\mathbf{u}'(t)$

Example 3: Find the domain of both $\overrightarrow{r(t)}$ and $\overrightarrow{r'(t)}$ if $\overrightarrow{r(t)} = \left\langle t\cos t, \sqrt{\ln t}, \frac{t}{t-2}\right\rangle$.

$(\ln t)^{\frac{1}{2}}$

all t $t \geq 1$ $t \neq 2$





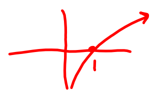
$1 \leq t < 2,\ t > 2$

$\boxed{\text{domain } \overrightarrow{r(t)}: [1,2)\cup(2,\infty)}$

$r'(t) = \left\langle t(-\sin t) + \cos t, \frac{1}{2}(\ln t)^{-\frac{1}{2}}\left(\frac{1}{t}\right), \frac{(t-2)(1) - t(1)}{(t-2)^2}\right\rangle$

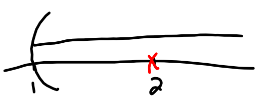
$t > 1$ only $t \neq 2$

$\overrightarrow{r'(t)} = \left\langle -t\sin t + \cos t, \frac{1}{2t\sqrt{\ln t}}, \frac{-2}{(t-2)^2}\right\rangle$



if $t = 1$,
$\ln(1) =0$
divide by 0

domain $\overrightarrow{r'(t)}$: $(1,2)\cup(2,\infty)$



denoted by $T(t)$

Example 4: Find the **Unit Tangent Vector** (tangent vector of length one) to the curve $\overrightarrow{r(t)} = \langle e^{2t}, t, te^t \rangle$ at $t = 0$.

Def tangent vector of $\overrightarrow{r(t)}$ is $\overrightarrow{r'(t)}$

$$T(t) = \frac{r'(t)}{|r'(t)|}$$

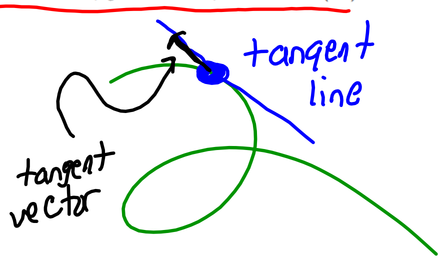
$$\overrightarrow{r'(t)} = \langle 2e^{2t}, 1, te^t + e^t \rangle$$

$$r'(0) = \langle 2, 1, 1 \rangle$$

$$\boxed{T(0) = \frac{\langle 2, 1, 1 \rangle}{\sqrt{6}}}$$

Example 5: Find parametric equations for the tangent line to the curve $x = 2\cos t$, $y = 2\sin t$, $z = 4\cos(2t)$ at the point $(\sqrt{3}, 1, 2)$. ← r_0

$$\overrightarrow{r(t)} = \langle 2\cos t, 2\sin t, 4\cos(2t) \rangle$$



equation of a line in space is $r(t) = \vec{r}_0 + t\vec{v}$

$\vec{v} = r'(?)$

$?$: solve $r(t) = \langle \sqrt{3}, 1, 2 \rangle$ for t

$$r'(t) = \langle -2\sin t, 2\cos t, -8\sin(2t) \rangle$$

$2\cos t = \sqrt{3}$
$2\sin t = 1$
$4\cos(2t) = 2$

$t = \frac{\pi}{6}$

$$\vec{v} = r'\left(\tfrac{\pi}{6}\right) = \left\langle -1, \sqrt{3}, -8\tfrac{\sqrt{3}}{2} \right\rangle$$

$$\vec{v} = \langle -1, \sqrt{3}, -4\sqrt{3} \rangle$$

$$\vec{r}_0 + t\vec{v} = \langle \sqrt{3}, 1, 2 \rangle + t\langle -1, \sqrt{3}, -4\sqrt{3} \rangle$$

$$\boxed{\begin{aligned} x &= \sqrt{3} - t \\ y &= 1 + \sqrt{3}t \\ z &= 2 - 4\sqrt{3}t \end{aligned}}$$

Definition: If $\overrightarrow{r(t)} = \langle f(t), g(t), h(t)\rangle$, where f, g and h are integratible functions, then

(i) $\int \overrightarrow{r(t)}\,dt = \left\langle \int f(t)\,dt, \int g(t)\,dt, \int h(t)\,dt \right\rangle$

(ii) $\int_a^b \overrightarrow{r(t)}\,dt = \left\langle \int_a^b f(t)\,dt, \int_a^b g(t)\,dt, \int_a^b h(t)\,dt \right\rangle$

Example 6: Find $\int \left(te^{3t}\mathbf{i} + t\ln t\mathbf{j} + t\cos(t^2)\mathbf{k} \right)\, dt$

parts parts u-sub

$u = t^2$

$\frac{1}{2}\sin(t^2) + C_3$

deriv integrate

$$\begin{array}{c|c} u & dv \\ \hline t \; (+) & e^{3t} \\ 1 \; (-) & \frac{1}{3}e^{3t} \\ 0 & \frac{1}{9}e^{3t} \end{array}$$

$\int te^{3t}\,dt$

$\frac{t}{3}e^{3t} - \frac{1}{9}e^{3t} + C_1$

$$\begin{array}{c|c} u & dv \\ \hline \ln t & t \\ \frac{1}{t} \; -\!\int & \frac{t^2}{2} \end{array}$$

$\int t\ln t\,dt$

$\frac{t^2}{2}\ln t - \int \frac{t}{2}\,dt$

$\frac{t^2}{2}\ln t - \frac{t^2}{4} + C_2$

$\left\langle \frac{t}{3}e^{3t} - \frac{1}{9}e^{3t} C_1, \frac{t^2}{2}\ln t - \frac{t^2}{4} + C_2, \frac{1}{2}\sin(t^2) + C_3 \right\rangle$

Example 7: Find $\overrightarrow{r(t)}$ if $\overrightarrow{r'(t)} = \left\langle \sin t, e^{-2t}, \frac{1}{t^2+1} \right\rangle$ and $\overrightarrow{r(0)} = \langle 2, 1, -2\rangle$.

$r(t) = \int r'(t)\,dt$

$= \left\langle -\cos t + C_1, -\frac{1}{2}e^{-2t} + C_2, \arctan t + C_3 \right\rangle$

To find C_1, C_2, C_3:

$-\cos(0) + C_1 = 2$

$\boxed{C_1 = 3}$

$-\frac{1}{2}e^0 + C_2 = 1$

$\boxed{C_2 = 1 + \frac{1}{2} = \frac{3}{2}}$

$\arctan(0) + C_3 = -2$

$\boxed{C_3 = -2}$

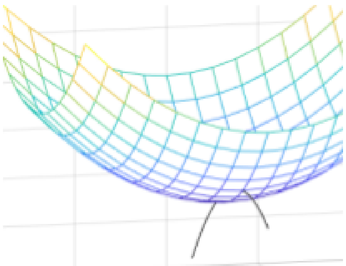
$\boxed{\left\langle -\cos t + 3, -\frac{1}{2}e^{-2t} + \frac{3}{2}, \arctan t - 2 \right\rangle}$

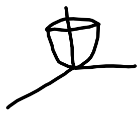
Example 8: Evaluate $\int_0^1 \left(t^2\mathbf{i} + t\mathbf{j} + \frac{1}{t+1}\mathbf{k}\right)\, dt$

Example 9: At what points does the curve $\overrightarrow{r(t)} = t\mathbf{i} + (2t - t^2)\mathbf{k}$ intersect the paraboloid $z = x^2 + y^2$?

space curve

surface





$x = t$

$y = 0$

$z = 2t - t^2$

$2t - t^2 = t^2 + 0^2$

$2t - 2t^2 = 0$

$2t(1 - t) = 0$

$$\boxed{\begin{array}{l} t = 0 \\ t = 1 \end{array}}$$

$$\boxed{\begin{array}{ll} t = 0 & (0, 0, 0) \\ t = 1 & (1, 0, 1) \end{array}}$$

Definition: The set C of all points $(f(t), g(t), h(t))$, where t varies over its domain, is called a **space curve.**

Example 10: Match the parametric equations with the graphs (labeled I-VI)

a. $x = t\cos t,\ y = t,\ z = t\sin t,\ t \geq 0$

b. $x = \cos t,\ y = \sin t,\ z = \dfrac{1}{1+t^2}$

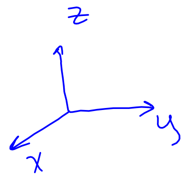
c. $x = t,\ y = \dfrac{1}{1+t^2},\ z = t^2$

d. $x = \cos t,\ y = \sin t,\ z = \cos(2t)$

e. $x = \cos 8t,\ y = \sin 8t,\ z = e^{0.8t}$

f. $x = \cos^2 t,\ y = \sin^2 t,\ z = t$

$\lim_{t\to\infty} \dfrac{1}{1+t^2} = 0$

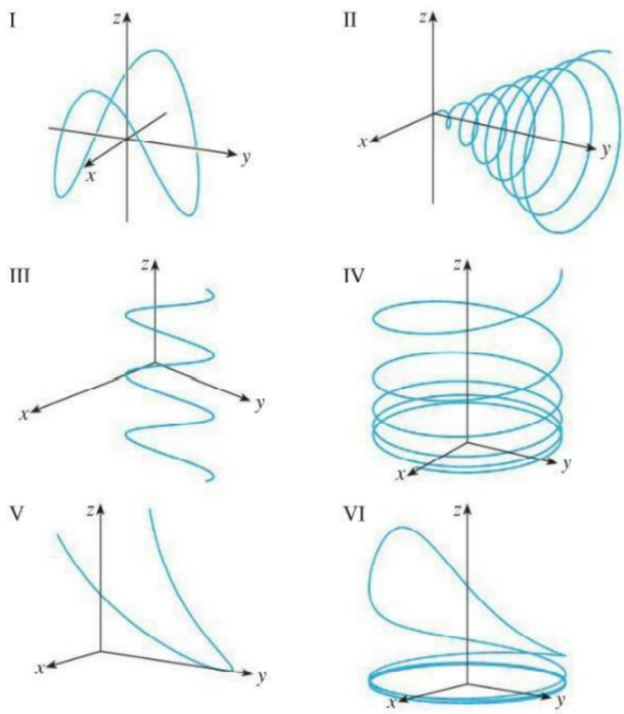


look at c

$\lim_{t\to\infty} t = \infty$

$\lim_{t\to\infty} \dfrac{1}{1+t^2} = 0 \to$ curve asymptotic towards xz plane

$\lim_{t\to\infty} t^2 = \infty$ always positive



$x = \cos t$

$y = \sin t$

$z = \cos(2t)$

$x^2 + y^2 = \cos^2 t + \sin^2 t = 1$

$x^2 + y^2 = 1$

Example ~~10~~ 11: At what point do the curves $\overrightarrow{r_1(t)} = \langle t, 1-t, 3+t^2 \rangle$ and $\overrightarrow{r_2(s)} = \langle 3-s, s-2, s^2 \rangle$ intersect? Find the angle of intersection correct to the nearest degree.

Do they intersect?

$\boxed{t = 3-s}$

$1 - t = s - 2 \rightarrow 1-3+s = s-2$

$-2 = -2$

no info

$3 + t^2 = s^2$

$3 + (3-s)^2 = s^2$

$3 + 9 - 6s + s^2 = s^2$

$12 = 6s$

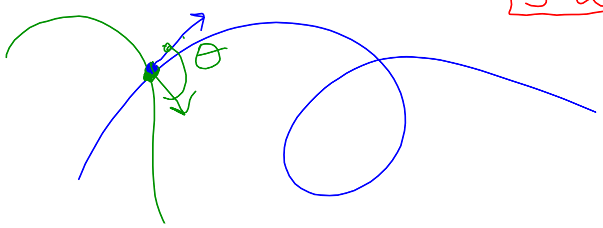
$\boxed{s = 2 \qquad t = 1}$ ← satisfies second equation

Point of intersection:

$\overrightarrow{r_1(1)} = \langle 1, 0, 4 \rangle$

$\overrightarrow{r_2(2)} = \langle 1, 0, 4 \rangle$

Point of intersection: $(1, 0, 4)$



angle between two tangent vectors at $t = 1 \quad s = 2$

Example 10: At what point do the curves $\overrightarrow{r_1(t)} = \langle t, 1-t, 3+t^2 \rangle$ and $\overrightarrow{r_2(s)} = \langle 3-s, s-2, s^2 \rangle$ intersect? Find the angle of intersection correct to the nearest degree.

Tangent vector for $r_1(t)$ at $t = 1$ is $r_1'(1)$.

$r_1'(t) = \langle 1, -1, 2t \rangle \rightarrow \boxed{r_1'(1) = \langle 1, -1, 2 \rangle}$

Tangent vector for $r_2(s)$ at $s = 2$ is

$r_2'(2)$

$r_2'(s) = \langle -1, 1, 2s \rangle$

$\boxed{r_2'(2) = \langle -1, 1, 4 \rangle}$

$$\cos\theta = \frac{\langle 1, -1, 2 \rangle \cdot \langle -1, 1, 4 \rangle}{\sqrt{1+1+4}\sqrt{1+1+16}}$$

$$\theta = \arccos\left(\frac{-1-1+8}{\sqrt{6}\sqrt{18}} \right)$$