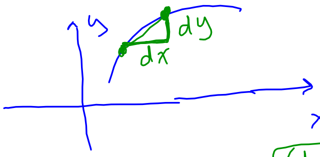
Section 13.3: Arc Length and Curvature

Recall from Math 152: If $x = f(t)$ and $y = g(t)$, then the length of the curve from $t = a$ to $t = b$ is given by $L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$. This can be extended to space curves, if $\overrightarrow{r(t)} = \langle f(t), g(t), h(t)\rangle$ and the curve is traversed exactly once as t increases from a to b, then the length of the curve is given by $L = \int_a^b \sqrt{(f'(t))^2 + (g'(t))^2 + (h'(t))^2}\, dt$. Putting this in more compact form,



$$\boxed{L = \int_a^b |\overrightarrow{r'(t)}|\, dt}$$

$$L \approx \sqrt{(dx)^2 + (dy)^2}$$

Example 1: Find the length of the helix with vector equation $\overrightarrow{r(t)} = \cos t\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k}$ from the point $(1, 0, 0)$ to the point $(1, 0, 2\pi)$.

$$L = \int_a^b |r'(t)|\, dt$$

Find a & b:

$\overrightarrow{r(0)} = \langle 1, 0, 0\rangle$

$r(2\pi) = \langle 1, 0, 2\pi\rangle$

$a = 0$

$b = 2\pi$

$$L = \int_0^{2\pi} |r'(t)|\, dt$$

$$r'(t) = \langle -\sin t, \cos t, 1\rangle$$

$$|r'(t)| = \sqrt{\underbrace{\sin^2 t + \cos^2 t}_{1} + 1}$$

$$= \sqrt{2}$$

$$L = \int_0^{2\pi} \sqrt{2}\, dt$$

$$= \boxed{2\pi\sqrt{2}}$$

Example 2: Find the length of the curve $\overrightarrow{r(t)} = \langle t^2, 2t, \ln t\rangle$ from $t = 1$ to $t = e$.

$$L = \int_1^e |r'(t)|\, dt \qquad r'(t) = \langle 2t, 2, \tfrac{1}{t}\rangle$$

$$L = \int_1^e \sqrt{4t^2 + 4 + \frac{1}{t^2}}\, dt$$

$$= \int_1^e \sqrt{\left(2t + \frac{1}{t}\right)^2}\, dt$$

$$= \int_1^e \left(2t + \frac{1}{t}\right) dt$$

$$\left(t^2 + \ln t\right)\Big|_1^e$$

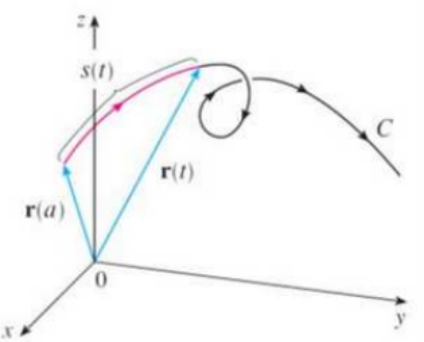
$$e^2 + \cancel{\ln e}^{1} - (1 + \cancel{\ln(1)}^{0})$$

$$\boxed{e^2}$$

Definition: Suppose C is a piecewise-smooth curve given by $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$, we define the **arc length function** s by

$$\boxed{s(t) = \int_a^t |\mathbf{r}'(\mathbf{u})|\, du} = \int_a^t \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2 + \left(\frac{dz}{du}\right)^2}\, du$$

Thus we can think of $s(t)$ as the length of the curve C between $\mathbf{r}(a)$ and $\mathbf{r}(t)$.



Example 3: Find the arc length function for $\overrightarrow{r(t)} = (5-t)\mathbf{i} + (4t-3)\mathbf{j} + 3t\mathbf{k}$ from the point $(4, 1, 3)$ in the direction of increasing t.

$\overrightarrow{r(u)} = \langle 5-u, 4u-3, 3u \rangle$

$r'(u) = \langle -1, 4, 3 \rangle$

$a = 1$

since $r(1) = \langle 4, 1, 3 \rangle$

$s(t) = \int_1^t |r'(u)|\, du$

$s(t) = \int_1^t \sqrt{1+16+9}\, du$

$s(t) = \int_1^t \sqrt{26}\, du$

$= \sqrt{26}\, u \Big|_1^t$

$\boxed{\sqrt{26}\,(t-1) = s(t)}$

$s(t) = \sqrt{26}\, t - \sqrt{26}$

Parameterizations: A single curve C can be represented by more than one vector function. For example, $\mathbf{r}(t) = \langle t, t^2, t^3 \rangle$, $0 \le t \le 2$ could also be written as $\mathbf{r}(u) = \langle 2u, 4u^2, 8u^3 \rangle$, $0 \le u \le 1$. We call this a **reparameterization** of the curve C. Moreover, it can be shown that arc length is independent of the parameterization used.

Example 4: Reparameterize the helix $\mathbf{r}(t) = \langle \cos t, \sin t, t \rangle$ with repect to arc length measured from $(1, 0, 0)$ in the direction of increasing t.

arc length function is

$$\boxed{s(t) = \int_a^t |r'(u)|\,du}$$

$a = 0$

$r'(u) = \langle -\sin u, \cos u, 1 \rangle$

$|r'(u)| = \sqrt{\sin^2 u + \cos^2 u + 1}$

$= \sqrt{2}$

$$s(t) = \int_0^t \sqrt{2}\,du$$

$$= \sqrt{2}\,u \Big|_0^t$$

$$= \sqrt{2}\,t - \sqrt{2}(0)$$

$$s(t) = \sqrt{2}\,t$$

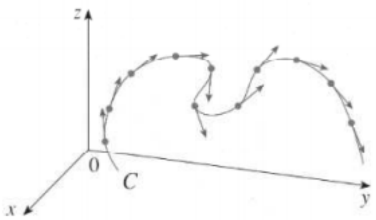
$$t = \frac{s}{\sqrt{2}}$$

s = arclength function

$$\boxed{r(s) = \left\langle \cos\frac{s}{\sqrt{2}}, \sin\frac{s}{\sqrt{2}}, \frac{s}{\sqrt{2}} \right\rangle}$$

Definition: The **curvature** of a curve C at a given point is a measure of how quickly a curve changes direction at that point. Specifically, we define it to be the magnitude of the rate of change of the unit tangent vector with respect to arc length (so the curvature will be independent of the parameterization). Thus the curvature of a curve is $\kappa = \left|\frac{d\mathbf{T}}{ds}\right|$. The curvature is easier to compute if it is expressed in terms of t instead of s, so we use the chain rule to write

$$\frac{d\mathbf{T}}{dt} = \frac{d\mathbf{T}}{ds}\frac{ds}{dt} \text{ and } \kappa = \left|\frac{d\mathbf{T}}{ds}\right| = \left|\frac{d\mathbf{T}/dt}{ds/dt}\right| \text{ but } ds/dt = |\mathbf{r}'(t)|, \boxed{\text{so } \kappa(t) = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|}}$$



use for κ always

Another useful formula for the curvature of C given by $\mathbf{r}(t)$ is $\boxed{\kappa(t) = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}}$.

Example 5: Find the curvature of $\mathbf{r}(t) = \langle \cos(3t), t, \sin(3t) \rangle$.

$$K = \frac{|r' \times r''|}{|r'|^3}$$

$$r' = \langle -3\sin(3t), 1, 3\cos(3t) \rangle$$

$$|r'| = \sqrt{9\sin^2(3t) + 1 + 9\cos^2(3t)}$$

$$= \sqrt{10} \quad \text{so} \quad \boxed{|r'|^3 = 10\sqrt{10}}$$

$$r'' = \langle -9\cos(3t), 0, -9\sin(3t) \rangle$$

$$r' \times r'' = \begin{vmatrix} i & j & k \\ -3\sin(3t) & 1 & 3\cos(3t) \\ -9\cos(3t) & 0 & -9\sin(3t) \end{vmatrix}$$

$$= \langle -9\sin(3t), -(\underbrace{27\sin^2 3t + 27\cos^2(3t)}_{27}), 9\cos(3t) \rangle$$

$$= \langle -9\sin(3t), -27, 9\cos(3t) \rangle$$

$$|r' \times r''| = \sqrt{81 + (27)^2}$$

$$= \sqrt{810} = 9\sqrt{10}$$

$$K = \frac{9\sqrt{10}}{10\sqrt{10}} = \boxed{\frac{9}{10}}$$

Example 6: Find the curvature of $\mathbf{r}(t) = \langle 1+t, 1-t, 3t^2 \rangle$. at the point $(2, 0, 3)$.

$$K = \frac{|r' \times r''|}{|r'|^3}$$

Find K at $t = 1$

$r'(t) = \langle 1, -1, 6t \rangle$ $\quad r'(1) = \langle 1, -1, 6 \rangle$ $\quad |r'(1)| = \sqrt{38}$

$r''(t) = \langle 0, 0, 6 \rangle$ $\quad r''(1) = \langle 0, 0, 6 \rangle$ $\quad |r'(1)|^3 = 38\sqrt{38}$

cross!

$$r'(1) \times r''(1) = \begin{vmatrix} i & j & k \\ 1 & -1 & 6 \\ 0 & 0 & 6 \end{vmatrix} = \langle -6, -6, 0 \rangle$$

$$|r' \times r''| = \sqrt{36 + 36 + 0} = \sqrt{72}$$

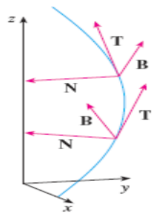
$$\boxed{K = \frac{\sqrt{72}}{38\sqrt{38}}}$$

Definition: Since $|\mathbf{T}| = 1$, a constant, we saw from section 11.6 this means $\mathbf{T}$ is perpendicular to $\mathbf{T}'$. We define the **principal unit normal vector** as $\mathbf{N}(t) = \dfrac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}$.

$x^2 + z^2 = 1$

space curve lies on the cylinder

Example 7: Find the unit tangent and unit normal vectors for $\mathbf{r}(t) = \langle \sin t, 2t, \cos t\rangle$.



$$T(t) = \frac{r'(t)}{|r'(t)|}$$

$$r'(t) = \langle \cos t, 2, -\sin t\rangle$$

$$|r'(t)| = \sqrt{\cos^2 t + 4 + \sin^2 t}$$

$$|r'(t)| = \sqrt{5}$$

unit tangent is $T(t) = \frac{1}{\sqrt{5}}\langle \cos t, 2, -\sin t\rangle$

unit normal vector is $N(t) = \dfrac{T'}{|T'|}$

$|ca| = c|a|$

c = constant

$$T'(t) = \frac{1}{\sqrt{5}}\langle -\sin t, 0, -\cos t\rangle$$

$$|T'(t)| = \frac{1}{\sqrt{5}}\sqrt{1}$$

$$= \frac{1}{\sqrt{5}}$$

$\boxed{N(t) = \langle -\sin t, 0, -\cos t\rangle}$

$$N(t) = \frac{T'(t)}{|T'(t)|} = \frac{\cancel{\frac{1}{\sqrt{5}}}\langle -\sin t, 0, -\cos t\rangle}{\cancel{\frac{1}{\sqrt{5}}}}$$