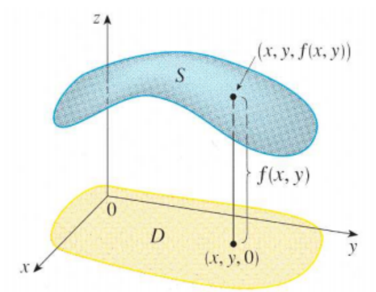
Section 14.1 Functions of Several Variables

Definition: Let D be a subset of R^2. A **function** f **of two variables** is a rule that assigns to each ordered pair (x, y) in D a unique real number $f(x, y)$. The set D is called the **domain** of f and its **range** is the set of values $\{f(x, y)|(x, y) \in D\}$. Note: Just as the domain of $y = f(x)$ is the set of all values of x for which $f(x)$ is defined, the domain of $\boxed{z = f(x, y)}$ is the set of all **points** (x, y) for which $f(x, y)$ is defined. **Thus the domain of** $z = f(x, y)$ **lies entirely in the** xy**-plane.**



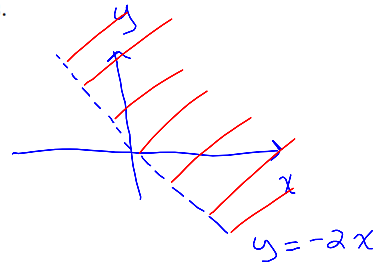
$z = f(x, y)$

Example 1: Find and sketch the domain of the following functions.

a.) $f(x, y) = \ln(2x + y)$

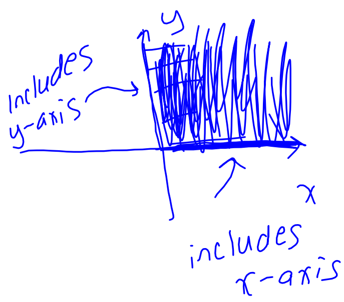
$2x + y > 0$

$y > -2x$



b.) $f(x, y) = \sqrt{x} + \sqrt{y}$

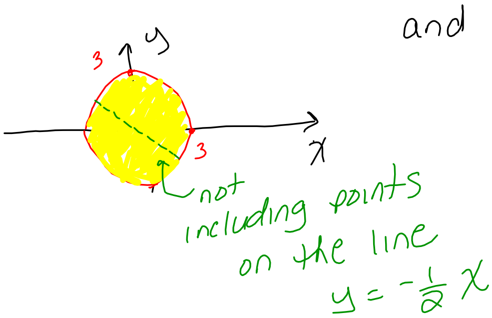
$x \geq 0$ and $y \geq 0$



c.) $f(x, y) = \dfrac{\sqrt{9 - x^2 - y^2}}{x + 2y}$

$9 - x^2 - y^2 \geq 0$

and $x + 2y \neq 0 \rightarrow y \neq -\frac{1}{2}x$

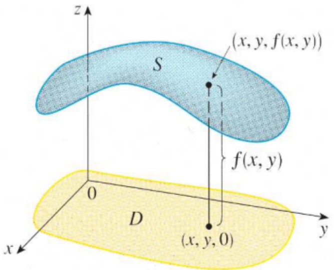


$9 - x^2 - y^2 \geq 0$

$9 \geq x^2 + y^2$ on and inside $x^2 + y^2 = 3$

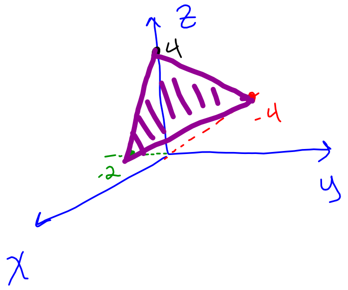
Definition: If f is a function of two variables with domain D, the **graph** of f is the set

$S = \{(x, y, z) \in R^3 | z = f(x, y), (x, y) \in D\}$. Just as the graph of a function of one variable is a curve with equation $y = f(x)$, the graph of a function f of two variables is a **surface** S with equation $\boxed{z = f(x, y)}$. We can visualize the graph S of f as lying directly above or below its domain D in the xy plane.

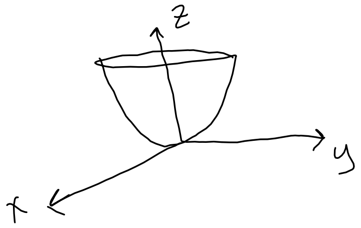


Example 2: Sketch the graph of:

a.) $f(x, y) = x + 2y + 4$ $\quad z = x + 2y + 4 \quad$ plane!

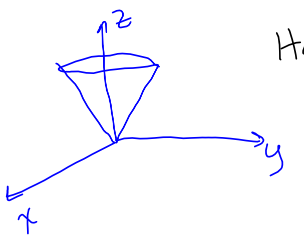


b.) $f(x, y) = x^2 + 9y^2$ $\quad z = x^2 + 9y^2 \quad$ elliptic paraboloid

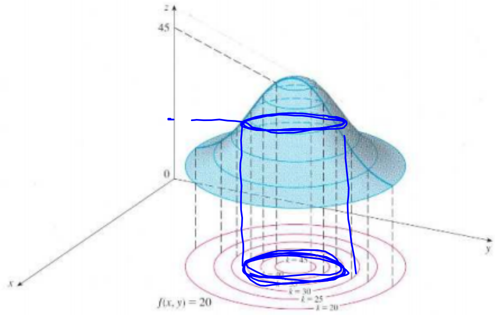


c.) $f(x, y) = \sqrt{x^2 + y^2}$ $\quad z = \sqrt{x^2 + y^2} \qquad z^2 = x^2 + y^2$

Half cone $\qquad$ cone



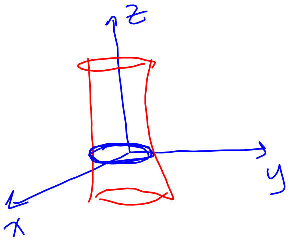
Definition: The **level curves** of a function f of two variables are the curves with equations $f(x, y)$ where k is a constant in the range of f. In other words, a level curve shows where the graph of f has h k. The level curves $f(x, y)$ are just the horizontal traces of the graph of f in the plane $z = k$ projected to the xy plane. A graph of the level curves is called a **countour plot**.



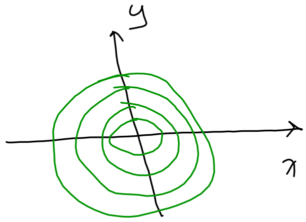
Example 3: Describe the level curves of

a.) a hyperboloid of one sheet

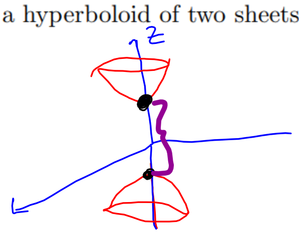
$x^2 + y^2 - z^2 = 1$

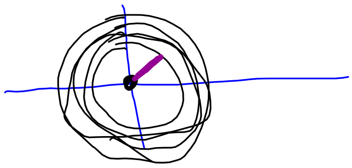


Family of ellipses



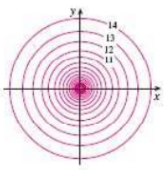
b.) a hyperboloid of two sheets

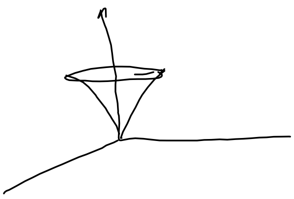




$-x^2 - y^2 + z^2 = 1$

Example 4: For the level curve shown below, sketch the surface.





Example 5: Sketch the level curves for the following functions:

level curve are of the form $z=k$

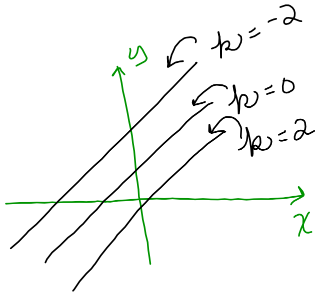
a.) $f(x,y) = 2 + 4x - y$ for $\boxed{k = -2, 0, 2.}$

$z = 2 + 4x - y$ surface is a plane

$k=-2$: $z=-2 \rightarrow -2 = 2 + 4x - y \rightarrow y = 4x+4$

$k=0$ $z=0 \rightarrow 0 = 2 + 4x - y \rightarrow y = 4x+2$

$k=2$ $z=2 \rightarrow 2 = 2 + 4x - y \rightarrow y = 4x$



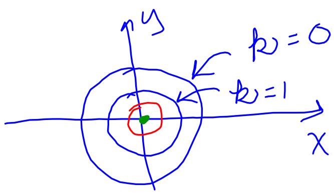
b.) $f(x,y) = \sqrt{9 - x^2 - y^2}$ for $k = 0, 1, 2, 3.$

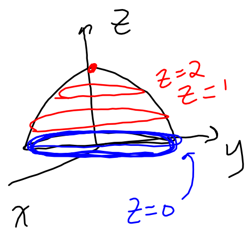
$z = \sqrt{9 - x^2 - y^2}$

$k=0$: $z=0$

$0 = \sqrt{9-x^2-y^2}$

$x^2 + y^2 = 9$





$k=1$: $1 = \sqrt{9-x^2-y^2}$

$1 = 9 - x^2 - y^2$

$x^2 + y^2 = 8$

$k=2$: $2 = \sqrt{9-x^2-y^2}$

$x^2 + y^2 = 5$

$k=3$: $3 = \sqrt{9-x^2-y^2}$

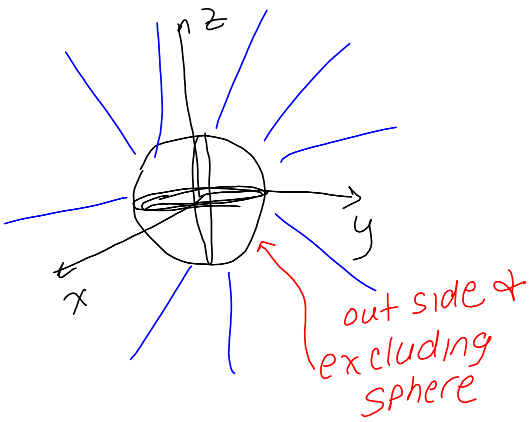
$x^2 + y^2 = 0$

Definition: A **function of three variables**, f, is a rule that assigns to each ordered triple (x, y, z) in the doman $D \subset R^3$ a real number $w = f(x, y, z)$. The graph of a function of three variables is in four dimensional space. The domain of a function of three variables is in three dimension.

Example 6: Find the domain of $f(x, y, z) = \dfrac{1}{\sqrt{x^2 + y^2 + z^2 - 9}}$.

$x^2 + y^2 + z^2 - 9 > 0$

$x^2 + y^2 + z^2 > 9$



A function of three variables has **level surfaces**, which are surfaces of the form $f(x, y, z) = k$. If the point $f(x, y, z)$ moves along a level surface, the value $f(x, y, z)$ remains fixed.

Example 7: Describe the level surfaces of $\boxed{f(x, y, z) = x + y + z.}$

Let $w = x + y + z$

level surfaces are of the form fix $w = k$

ex: $w = 2 \rightarrow k = 2$

$2 = x + y + z$ plane!

level surfaces are families of planes

Example 8: Describe the level surfaces of $f(x, y, z) = x^2 + y^2 + z^2$.

let $w = x^2 + y^2 + z^2$

Level surfaces are families of spheres.

for example if $w = 2$

$2 = x^2 + y^2 + z^2$