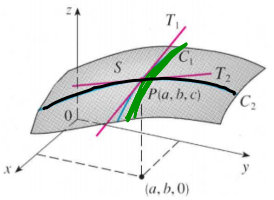
Section 14.3 Partial Derivatives

If f is a function of two variables x and y, suppose we only let x vary while keeping y fixed, say $y = b$, where b is constant. Then we are really considering a function of a single variable x, namely $g(x) = f(x, b)$. If g has a derivative ar $x = a$, then we call it the **partial derivative of** f **respect to** x **at the point** (a, b), denoted by $f_x(a, b)$. Similarly, the **partial derivative of** f **respect to** y **at the point** (a, b), denoted by $f_y(a, b)$, is obtained by letting y vary while keeping x fixed.



Example 1: If $f(x, y) = x^2 + 2y^3 - 4xy$, find $f_x(-1, 2)$ and $f_y(-1, 2)$.

y constant $f_x(x, y) = 2x + 0 - 4y$

$f_x(-1, 2) = -2 - 8 = -10$

x constant $f_y(x, y) = 0 + 6y^2 - 4x$

$f_y(-1, 2) = 24 + 4 = 28$

Example 2: If $f(x, y) = ye^{-x} + 2x$, find $\left.\frac{\partial f}{\partial x}\right|_{(1,0)}$ and $\left.\frac{\partial f}{\partial y}\right|_{(1,0)}$

$f_x = \frac{\partial f}{\partial x} = y(-e^{-x}) + 2$

$\left.\frac{\partial f}{\partial x}\right|_{(1,0)} = 0 + 2 = 2$

$f_y = \frac{\partial f}{\partial y} = e^{-x}$

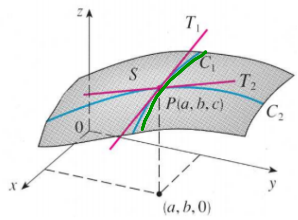
$\left.\frac{\partial f}{\partial y}\right|_{(1,0)} = e^{-1}$

Example 3: Find $f_x(x, y)$ and $f_y(x, y)$ for $f(x, y) = e^{\sin(2xy)}$.

$f_x(x, y) = \left(e^{\sin(2xy)}\right)\cos(2xy)(2y)$

$f_y(x, y) = e^{\sin(2xy)} \cos(2xy)(2x)$

Geometric interpretation of partial derivatives: Recall that the equation $z = f(x,y)$ represents a surface S. If $f(a,b) = c$, then the point $P(a,b,c)$ lies on S. The vertical plane $y = b$ intersects S in a curve C_1. Similarly, The vertical plane $x = a$ intersects S in a curve C_2. Note that the curve C_1 is the graph of $g(x) = f(x,b)$, so the slope of the tangent line to the graph of $g(x)$ at the point P is $g'(a) = f_x(a,b)$, Similarly, the curve C_2 is the graph of $h(y) = f(a,y)$, so the slope of the tangent line to the graph of $h(y)$ at the point P is $h'(b) = f_y(a,b)$. **Thus, the partial derivatives $f_x(a,b)$ and $f_y(a,b)$ can be interpreted geometrically as the slopes of the tangent lines at the point $P(a,b,c)$ to the traces C_1 and C_2 of S in the planes $y = b$ and $x = a$.**



$f_x(a,b) =$ slope of the curve C_1

$f_y(a,b) =$ slope of the curve C_2

Functions of more than two variables: Partial derivatives can also be defined for functions of three variables, for example if f is a function of x, y and z, then the partial derivative of f with respect to x is found by holding y and z constant and differentiating f with respect to x.

Example 4: Find f_x, f_y and f_z for:

(i) $f(x,y,z) = xy^2z^3$

$$f_x(x,y,z) = y^2z^3$$

$$f_y(x,y,z) = 2xyz^3$$

$$f_z(x,y,z) = 3xy^2z^2$$

(ii) $f(x,y,z) = xe^y + ye^z + ze^x$

$$f_x = e^y + ze^x$$

$$f_y = xe^y + e^z$$

$$f_z = ye^z + e^x$$

Higher derivatives: If f is a function of two variables, then its partial derivatives f_x and f_y are also functions of two variables, so we consider their partial derivatives $(f_x)_x$, $(f_x)_y$, $(f_y)_x$ and $(f_y)_y$. These are called the **second order partial derivatives** of f. If $x = f(x, y)$, we use the following notation:

(i) $f_{xx} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial x}\right)$

(ii) $f_{xy} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial x}\right)$

(iii) $f_{yx} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial y}\right)$

(iv) $f_{yy} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial y}\right)$

Clairaut's Theorem: If f is defined on a disk D that containes the point (a, b), and f_{xy} and f_{yx} are continuous on D, then $f_{xy}(a, b) = f_{yx}(a, b)$.

Example 5: Find all second order partial derivatives for $f(x, y) = x\cos y + ye^x$

$$f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$$

$$= \frac{\partial}{\partial x}\left(\cos y + ye^x\right)$$

$$\boxed{f_{xx} = ye^x}$$

$$f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}\left(\cos y + ye^x\right)$$

$f_{xy} = -\sin y + e^x$

$f_{yx} = -\sin y + e^x$

since $f_{yx} = f_{xy}$

f_{yy} you try on your own.