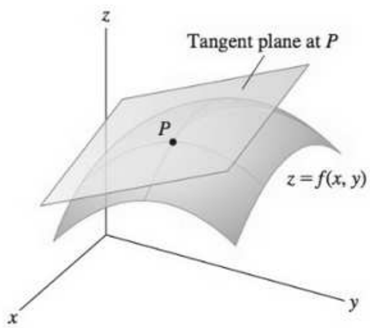
Section 14.4 Tangent Planes and Differentials

One of the most significant ideas in single variable calculus is how the tangent line can be used to approximate a function at its point of tangency. If we zoom in toward a point on a graph of a differentiable function, the graph becomes indistinguishable from its tangent line. Likewise, a tangent plane can be used to approximate a surface at a point. We need to develop a formula for the equation of a tangent plane to a surface $z = f(x, y)$ at a point (x_0, y_0, z_0).

Suppose $z = f(x, y)$ is a surface and let $P_0(x_0, y_0, z_0)$ be a point on the surface. Let C_1 be the curve obtained by intersecting the surface with the plane $y = y_0$. Then the slope of the tangent line to the curve C_1 at the point P is $f_x(x_0, y_0)$. Similarly, Let C_2 be the curve obtained by intersecting the surface with the plane $x = x_0$. Then the slope of the tangent line to the curve C_2 at the point P is $f_y(x_0, y_0)$. The **tangent plane** is defined to be the plane that contains both of these tangent lines.



Recall the equation of a plane is $\mathbf{n} \cdot (\mathbf{r} - \mathbf{r_0}) = 0$. If we can find the direction vector of each tangent line, then we know from a previous result that the normal vector to the plane is the cross product of these two direction vectors.

Let $\mathbf{r}(x, y) = \langle x, y, f(x, y)\rangle$. Then $\mathbf{r}_x(x_0, y_0) = \langle 1, 0, f_x(x_0, y_0)\rangle$ and $\mathbf{r}_y(x_0, y_0) = \langle 0, 1, f_y(x_0, y_0)\rangle$ are the two tangent vectors to the curves C_1 and C_2, respectively. Since the tangent plane contains both tangent vectors, $\mathbf{n} = \mathbf{r}_x(x_0, y_0) \times \mathbf{r}_y(x_0, y_0)$, hence

$\mathbf{n} = \langle -f_x(x_0, y_0), -f_y(x_0, y_0, 1\rangle$. The tangent plane would then be $\mathbf{n} \cdot (\mathbf{r} - \mathbf{r_0}) = 0$, which yields

$\langle -f_x(x_0, y_0), -f_y(x_0, y_0, 1\rangle \cdot \langle x - x_0, y - y_0, z - z_0\rangle = 0$ and hence

$-f_x(x_0, y_0)(x-x_0)-f_y(x_0, y_0)(y-y_0)+z-z_0 = 0$, and therefore $z-z_0 = f_x(x_0, y_0)(x-x_0)+f_y(x_0, y_0)(y-y_0)$.

Definition: The **tangent plane** to the surface $z = f(x, y)$ at the point (x_0, y_0, z_0) is

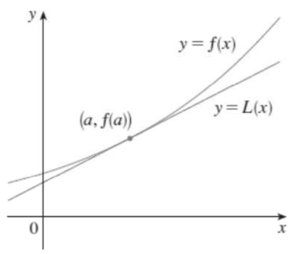
$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

Example 1: Find the equation of the tangent plane to the surface $z = x^2 + 3y^2$ at the point $(1, -1, 4)$.

Example 2: (a) Find the equation of the tangent plane to the surface $z = \ln(x - 3y)$ at the point $(7, 2, 0)$.

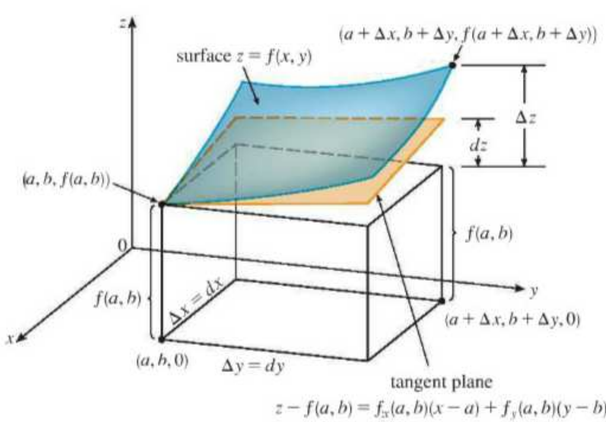
(b) Find the equation of the normal line to the plane at the point $(7, 2, 0)$.

Differentials: Recall from calculus 1 that if $y = f(x)$, then dx is an independent variable. The **differential** dy depends on dx and is defined as $dy = f'(x)dx$. Differentials can be used in estimating the errors that occur because of approximate measurements. This is seen in the use of the tangent line as an approximation to $f(x)$ near its point of tangency.



Example 3: Use differentials to approximate $(1.97)^6$.

Definition: Suppose $z = f(x, y)$, x changes from x to $x + \Delta x$ and y changes from y to $y + \Delta y$. Then the corresponding change in z is $\Delta z = f(x + \Delta x, y + \Delta y) - f(x, y)$. This represents the change in the height of the surface. The **differential** is defined as $dz = f_x(x, y)dx + f_y(x, y)dy$. This represents the change in height of the tangent plane.



Example 4: Find the differential of $z = y\sin(xy)$.

Example 5: Find the differential of $w = xyz$.

How we use differentials with functions of two variables: The same way we did for functions of one variable, except instead of using the tangent line we will be using the tangent plane.

$$z = z_0 + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

Example 6: Use differentials to approximate $\sqrt{9(1.95)^2 + (8.1)^2}$.

Example 7: Find the equation of the tangent plane to the surface $f(x, y) = xe^{xy}$ at the point $(1, 0, 1)$ and use this plane to approximate $f(1.1, -0.2)$.

Example 8: The base radius and height of a right circular cone are measured as 10 cm and 25 cm, respectively, with a possible error in measurement of 0.1 cm and 0.2 cm, respectively . Use differentials to estimate the maximum error in the calculated volume of the cone.