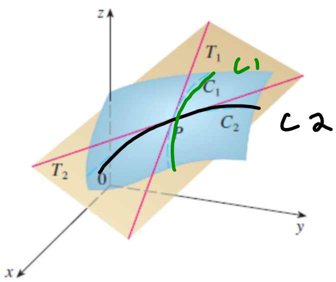
Section 14.4 Tangent Planes and Differentials

One of the most significant ideas in single variable calculus is how the tangent line can be used to approximate a function at its point of tangency. If we zoom in toward a point on a graph of a differentiable function, the graph becomes indistinguishable from its tangent line. Likewise, a tangent plane can be used to approximate a surface at a point. We need to develop a formula for the equation of a tangent plane to a surface $z = f(x, y)$ at a point (x_0, y_0, z_0).

Suppose $z = f(x, y)$ is a surface and let $P_0(x_0, y_0, z_0)$ be a point on the surface. Let C_1 be the curve obtained by intersecting the surface with the plane $y = y_0$. Then the slope of the tangent line to the curve C_1 at the point P is $f_x(x_0, y_0)$. Similarly, Let C_2 be the curve obtained by intersecting the surface with the plane $x = x_0$. Then the slope of the tangent line to the curve C_2 at the point P is $f_y(x_0, y_0)$. The **tangent plane** is defined to be the plane that contains both of these tangent lines.



Recall the equation of a plane is $\mathbf{n} \cdot (\mathbf{r} - \mathbf{r_0}) = 0$. If we can find the direction vector of each tangent line, then we know from a previous result that the normal vector to the plane is the cross product of these two direction vectors.

Let $\mathbf{r}(x, y) = \langle x, y, f(x, y)\rangle$. Then $\mathbf{r}_x(x_0, y_0) = \langle 1, 0, f_x(x_0, y_0)\rangle$ and $\mathbf{r}_y(x_0, y_0) = \langle 0, 1, f_y(x_0, y_0)\rangle$ are the two tangent vectors to the curves C_1 and C_2, respectively. Since the tangent plane contains both tangent vectors, $\mathbf{n} = \mathbf{r}_x(x_0, y_0) \times \mathbf{r}_y(x_0, y_0)$, hence

$\mathbf{n} = \langle -f_x(x_0, y_0), -f_y(x_0, y_0, 1\rangle$. The tangent plane would then be $\mathbf{n} \cdot (\mathbf{r} - \mathbf{r_0}) = 0$, which yields

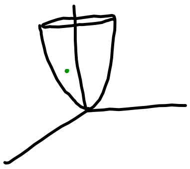
$\langle -f_x(x_0, y_0), -f_y(x_0, y_0, 1\rangle \cdot \langle x - x_0, y - y_0, z - z_0\rangle = 0$ and hence

$-f_x(x_0, y_0)(x-x_0)-f_y(x_0, y_0)(y-y_0)+z-z_0 = 0$, and therefore $z-z_0 = f_x(x_0, y_0)(x-x_0)+f_y(x_0, y_0)(y-y_0)$.

Definition: The **tangent plane** to the surface $z = f(x, y)$ at the point (x_0, y_0, z_0) is

$$\boxed{z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)}$$

Example 1: Find the equation of the tangent plane to the surface $z = x^2 + 3y^2$ at the point $(1, -1, 4)$.



$x_0 = 1,\ y_0 = -1,\ z_0 = 4$

$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$

$z - 4 = f_x(1, -1)(x - 1) + f_y(1, -1)(y + 1)$

$f(x, y) = x^2 + 3y^2$ $\qquad f_x(x, y) = 2x \qquad f_x(1, -1) = 2$

$f_y(x, y) = 6y \qquad f_y(1, -1) = -6$

$$\boxed{z - 4 = 2(x - 1) - 6(y + 1)}$$

Example 2: (a) Find the equation of the tangent plane to the surface $z = \ln(x - 3y)$ at the point $(7, 2, 0)$.

$x_0 = 7,\ y_0 = 2,\ z_0 = 0$

$z - 0 = f_x\cancel{(7, 2)}^{1}(x - 7) + f_y\cancel{(7, 2)}^{-3}(y - 2)$

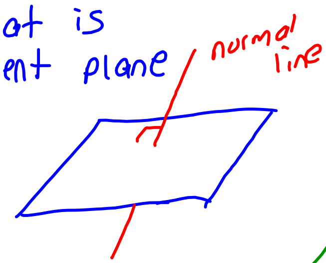
$f_x = \dfrac{1}{x - 3y} \qquad f_x(7, 2) = 1$

$f_y = \dfrac{-3}{x - 3y} \qquad f_y(7, 2) = -3$

$z = x - 7 - 3y + 6 \qquad \boxed{z = x - 3y - 1}$

(b) Find the equation of the normal line to the plane at the point $(7, 2, 0)$.

normal line is the line that is perpendicular to the tangent plane



$\vec{r}_0 + t\vec{v}$

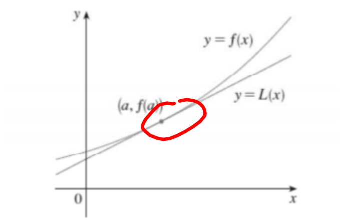
$\vec{v}$ = normal vector to the tangent plane

$z - x + 3y = -1$

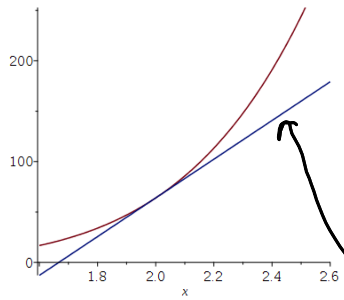
$-x + 3y + z = -1 \qquad \vec{n} = \langle -1, 3, 1 \rangle = \vec{v}$

$$\boxed{\langle 7, 2, 0 \rangle + t\langle -1, 3, 1 \rangle}$$

Differentials: Recall from calculus 1 that if $y = f(x)$, then dx is an independent variable. The **differential** dy depends on dx and is defined as $dy = f'(x)dx$. Differentials can be used in estimating the errors that occur because of approximate measurements. This is seen in the use of the tangent line as an approximation to $f(x)$ near its point of tangency.



Example 3: Use differentials to approximate $(1.97)^6$.



① tangent line to

$f(x) = x^6$ at $x = 2$

point: $(2, 64)$

$m = f'(2)$

$= 6(2)^5$

$\boxed{m = 192}$

$\boxed{y - 64 = 192(x-2)}$

$f'(x) = 6x^5$

$m = f'(2) = 192$

$y = 64 + 192(x-2)$

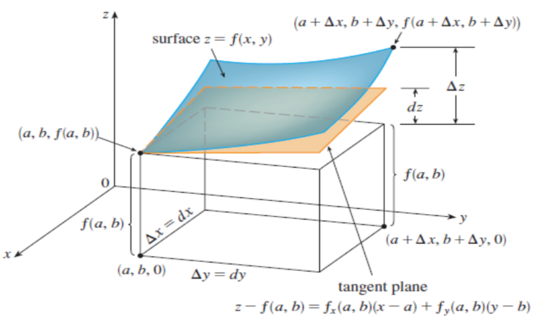
② use line at $x = 1.97$

$\boxed{y = 64 + \boxed{192(-0.03)}}$

differential

$\boxed{y = 58.24}$

Definition: Suppose $z = f(x, y)$, x changes from x to $x + \Delta x$ and y changes from y to $y + \Delta y$. Then the corresponding change in z is $\Delta z = f(x + \Delta x, y + \Delta y) - f(x, y)$. This represents the change in the height of the surface. The **differential** is defined as $dz = f_x(x, y)dx + f_y(x, y)dy$. This represents the change in height of the tangent plane.



Recall $y = f(x)$

$dy = f'(x)dx$

Example 4: Find the differential of $z = y\sin(xy)$.

$$dz = z_x\,dx + z_y\,dy$$

$$= y\left[\cos(xy)\,y\right]dx + \left[1\sin(xy) + y\cos(xy)\,x\right]dy$$

Example 5: Find the differential of $w = xyz$.

$$dw = w_x\,dx + w_y\,dy + w_z\,dz$$

$$= yz\,dx + xz\,dy + xy\,dz$$

How we use differentials with functions of two variables: The same way we did for functions of one variable, except instead of using the tangent line we will be using the tangent plane.

$$\boxed{z = z_0 + \underbrace{f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)}_{dz}}$$

Example 6: Use differentials to approximate $\sqrt{9(1.95)^2 + (8.1)^2}$.

$f(x,y) = \sqrt{9x^2 + y^2}$

Approximate $f(1.95, 8.1)$ by finding the tangent plane at $(2, 8, 10)$

$f(2,8) = \sqrt{9(4) + 64}$

$= \sqrt{100}$

$= 10$

Tangent plane: $z - 10 = f_x(2,8)(x-2) + f_y(2,8)(y-8)$

$f_x(x,y) = \frac{1}{2}(9x^2+y^2)^{-\frac{1}{2}}(18x) = \dfrac{9x}{\sqrt{9x^2+y^2}}$

$f_x(2,8) = \dfrac{18}{10} = \dfrac{9}{5}$

$f_y(x,y) = \frac{1}{2}(9x^2+y^2)^{-\frac{1}{2}}(2y) = \dfrac{y}{\sqrt{9x^2+y^2}}$

$f_y(2,8) = \dfrac{8}{10} = \dfrac{4}{5}$

$z - 10 = \frac{9}{5}(x-2) + \frac{4}{5}(y-8)$

$z = 10 + \frac{9}{5}(x-2) + \frac{4}{5}(y-8)$

$f(1.95, 8.1) \approx$ Tangent plane at $(2,8,10)$

$$\sqrt{9(1.95)^2+(8.1)^2} \approx 10 + \frac{9}{5}(1.95-2) + \frac{4}{5}(8.1-8)$$

$$\approx 10 + \frac{9}{5}(-0.05) + \frac{4}{5}(0.1)$$

$$\approx 9.99$$

Example 7: Find the equation of the tangent plane to the surface $f(x, y) = xe^{xy}$ at the point $\underline{(1, 0, 1)}$ and use this plane to approximate $f(1.1, -0.2)$. $x_0 = 1$, $y_0 = 0$, $z_0 = 1$

plane: $z - 1 = f_x(1,0)(x-1) + f_y(1,0)(y-0)$

$f(x,y) = xe^{xy}$

$f_x(x,y) = e^{xy} + xye^{xy}$ $\quad f_x(1,0) = 1$

$f_y(x,y) = x^2e^{xy}$ $\quad f_y(1,0) = 1$

$z - 1 = x - 1 + y - 0$

$\boxed{z = x + y}$

$f(1.1, -0.2) \approx 1.1 - 0.2 = .9$

Example 8: The base radius and height of a right circular cone are measured as 10 cm and 25 cm, respectively, with a possible error in measurement of 0.1 cm and 0.2 cm, respectively . Use differentials to estimate the maximum error in the calculated volume of the cone.

$\Delta V = V(10.1, 25.2) - V(10, 25)$

$V = \frac{1}{3}\pi r^2 h$, $\quad r = 10 \quad |dr| < .1$

$\Delta V = V(10, 25) - V(9.9, 24.8)$

$h = 25 \quad |dh| < .2$

max error in volume occurs if $dr = 0.1$
$dh = 0.2$

max error in volume is approximately dV

$V = \frac{1}{3}\pi r^2 h$ $\quad r = 10 \quad dr = 0.1$

$dV = V_r\,dr + V_h\,dh$ $\quad h = 25 \quad dh = 0.2$

$= \frac{2}{3}\pi rh\,dr + \frac{1}{3}\pi r^2\,dh$

$dV = \frac{2}{3}\pi(10)(25)(0.1) + \frac{1}{3}\pi(10)^2(0.2)$

$dV = \frac{70\pi}{3}\ \text{cm}^3$