Section 14.5 The Chain Rule

Recall the chain rule in 2d: If $y = f(x)$ and $x = g(t)$, then $\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$.

Example 1: If $y = x^3$ and $x = \cos t$, find $\frac{dy}{dt}$.

The chain rule in higher dimensions:

Chain rule, case 1: If $z = f(x, y)$, $x = g(t)$, $y = h(t)$, then

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

Example 2: If $z = 2x^2y^3 + 6xy^2$, $x = \frac{1}{t}$, $y = \cos t$, find $\frac{dz}{dt}$.

Example 3: If $z = \ln(x^2 + y^2)$, $x = 1 + e^{6t}$, $y = \sec^2(5t)$, find $\frac{dz}{dt}$

Chain rule, case 2: If $z = f(x, y)$, $x = g(s, t)$, $y = h(s, t)$, then

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \qquad \text{and} \qquad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

Example 4: if $z = x^2 - 3x^2y^3$, $x = se^t$, $y = se^{-t}$, find $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$.

Chain rule, general version: If $u = f(x_1, x_2, x_3, ..., x_n)$ and each x_i is a function of t_1, t_2, t_3,...,t_m, then

$$\frac{\partial u}{\partial t_i} = \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial t_i} + ... + \frac{\partial u}{\partial x_n}\frac{\partial x_n}{\partial t_i}$$

Example 5: If $u = x^4y + y^2z^3$, $x = rse^t$, $y = rs^2e^{-t}$, $z = r^2s\sin t$, find $\frac{\partial u}{\partial s}$, $\frac{\partial u}{\partial r}$, $\frac{\partial u}{\partial t}$.

Example 6: The radius of a circular cylinder is decreasing at a rate of 2 cm/s while the height is increasing at a rate of 5 cm/sec. At what rate is the volume of the cylinder changing when the radius is 80 cm and the height is 360 cm?

Example 7: The length l, width w and height h of a box change with time. At a certain instant, the dimensions are $l = 1$ m and $w = h = 2$ m, and l and w are increasing at rate of 2 m/s while h is decreasing at a rate of 3 m/s. At that same instant, find the rate at which the surface area is changing.