Section 14.5 The Chain Rule

Recall the chain rule in 2d: If $y = f(x)$ and $x = g(t)$, then $\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$.

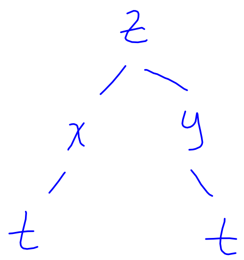
Example 1: If $y = x^3$ and $x = \cos t$, find $\frac{dy}{dt}$.

$$\frac{dy}{dt} = (3x^2)(-\sin t)$$

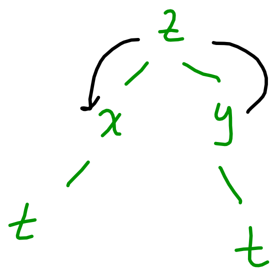
The chain rule in higher dimensions:

Chain rule, case 1: If $z = f(x, y)$, $x = g(t)$, $y = h(t)$, then

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$



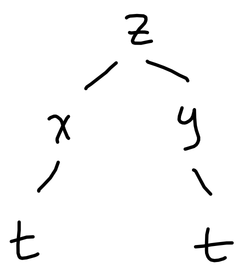
Example 2: If $z = 2x^2y^3 + 6xy^2$, $x = \frac{1}{t}$, $y = \cos t$, find $\frac{dz}{dt}$.



$$\frac{dz}{dt} = \underbrace{\frac{\partial z}{\partial x}}_{z_x}\frac{dx}{dt} + \underbrace{\frac{\partial z}{\partial y}}_{z_y}\frac{dy}{dt}$$

$$= (4xy^3 + 6y^2)\left(-\frac{1}{t^2}\right) + (6x^2y^2 + 12xy)(-\sin t)$$

Example 3: If $z = \ln(x^2 + y^2)$, $x = 1 + e^{6t}$, $y = \sec^2(5t)$, find $\frac{dz}{dt}$ =

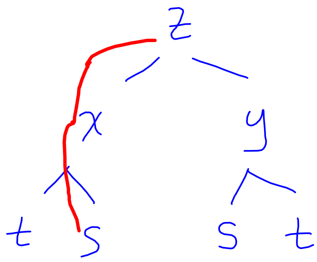


$$\frac{dz}{dt} = z_x\frac{dx}{dt} + z_y\frac{dy}{dt}$$

$$= \frac{2x}{x^2+y^2}\left(6e^{6t}\right) + \frac{2y}{x^2+y^2}\left(2\sec(5t)\cdot\sec(5t)\tan(5t)\cdot 5\right)$$

Chain rule, case 2: If $z = f(x,y)$, $x = g(s,t)$, $y = h(s,t)$, then

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \quad \text{and} \quad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$



Example 4: if $z = x^2 - 3x^2y^3$, $x = se^t$, $y = se^{-t}$, find $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$.

you do on your own later

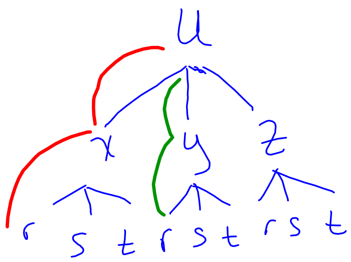
$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

$$= (2x - 6xy^3)(e^t) + (-9x^2y^2)(e^{-t})$$

Chain rule, general version: If $u = f(x_1, x_2, x_3, ..., x_n)$ and each x_i is a function of t_1, t_2, t_3,...,t_m, then

$$\frac{\partial u}{\partial t_i} = \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial t_i} + ... + \frac{\partial u}{\partial x_n}\frac{\partial x_n}{\partial t_i}$$

Example 5: If $u = x^4y + y^2z^3$, $x = rse^t$, $y = rs^2e^{-t}$, $z = r^2 s \sin t$, find $\frac{\partial u}{\partial s}$, $\frac{\partial u}{\partial r}$, $\frac{\partial u}{\partial t}$.



$$\boxed{\text{Quiz 3} \quad \text{Find } \frac{\partial u}{\partial t}}$$

NamE!

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial r}$$

$$= (4x^3y)(se^t) + (x^4 + 2yz^3)(s^2e^{-t}) + (3y^2z^2)(2rs\sin t)$$

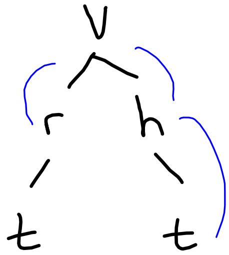
Example 6: The radius of a circular cylinder is decreasing at a rate of 2 cm/s while the height is increasing at a rate of 5 cm/sec. At what rate is the volume of the cylinder changing when the radius is 80 cm and the height is 360 cm?

$V(r,h) = \pi r^2 h$

Find $\left.\dfrac{dV}{dt}\right|_{\substack{r=80\\ h=360}}$

$\dfrac{dr}{dt} = -2$

$\dfrac{dh}{dt} = 5$



$$\frac{dV}{dt} = \frac{\partial V}{\partial r}\frac{dr}{dt} + \frac{\partial V}{\partial h}\frac{dh}{dt}$$
$$= 2\pi r h \frac{dr}{dt} + \pi r^2 \frac{dh}{dt}$$
$$= 2\pi(80)(360)(-2) + \pi(80)^2 5$$
$$= -83{,}200\pi \ \frac{\text{cm}^3}{\text{sec}}$$

Example 7: The length l, width w and height h of a box change with time. At a certain instant, the dimensions are $l = 1$ m and $w = h = 2$ m, and l and w are increasing at rate of 2 m/s while h is decreasing at a rate of 3 m/s. At that same instant, find the rate at which the surface area is changing.