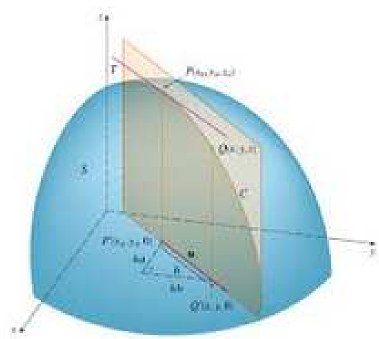
Section 14.6 Directional Derivatives

Recall: If $z = f(x,y)$, then $f_x(x_0, y_0)$ is the rate of change of z in the x-direction while y is held constant. Another way to view this is, $f_x(x_0, y_0)$ is the rate of change of z in the direction of the unit vector $\langle 1, 0\rangle$. Similarly, $f_y(x_0, y_0)$ is the rate of change of z in the direction of the unit vector $\langle 0, 1\rangle$.

Suppose now we wish to find the rate of change of z at (x_0, y_0) in the direction of an arbitrary unit vector $\mathbf{u} = \langle a, b\rangle$. To do this, we consider the surface S with equation $z = f(x,y)$ and let $z_0 = f(x_0, y_0)$. Then the point $P(x_0, y_0, z_0)$ lies on S. The vertical plane that passes thru P in the direction of $\mathbf{u}$ intersects the surface in a curve C. The slope of the tangent line T to C at P is the rate of change of z in the direction of $\mathbf{u}$, called the **directional derivative**.

Definition: The **directional derivative** of $z = f(x,y)$ at (x_0, y_0) in the direction of a unit vector $\mathbf{u} = \langle a, b\rangle$ is $D_{\mathbf{u}}f(x_0, y_0) = f_x(x_0, y_0)a + f_y(x_0, y_0)b = \langle f_x(x,y), f_y(x,y)\rangle \cdot \langle a, b\rangle$.



Example 1: Given $f(x,y) = x^3 - 4x^2y + y^2$, find the directional derivative at the point $(0, -1)$ in the direction $\mathbf{u} = \left\langle \frac{3}{5}, \frac{4}{5} \right\rangle$.

Example 2: Suppose $f(x,y) = y^2 + 2xy$. Find $D_{\mathbf{u}}f(x,y)$ at the point $(2, 3)$ where $\mathbf{u}$ is the unit vector corresponding to $\frac{\pi}{3}$.

Example 3: Find $D_{\mathbf{u}}f(x,y)$ at the point $(1,2)$ in the direction of $\langle 1,-3\rangle$ to the surface $f(x,y)=x^3+2x^2y^2$.

Example 4: If $f(x,y,z)=z^3-x^2y$, find $D_{\mathbf{u}}f(1,6,2)$ if $\mathbf{u}=\left\langle \frac{3}{13},\frac{4}{13},\frac{12}{13}\right\rangle$.

Definition: We define the **gradient** of $z=f(x,y)$ to be the vector $\nabla f=\langle f_x(x,y),f_y(x,y)\rangle$. Therefore, $D_{\mathbf{u}}f(x_0,y_0)=\langle f_x(x_0,y_0),f_y(x_0,y_0)\rangle\cdot\langle a,b\rangle=\nabla f\cdot\mathbf{u}$.

Example 5: If $f(x,y)=e^x-\cos(xy)$, find $\nabla f(1,0)$.

Fact: If we consider all possible directional derivatives at a given point, the **maximum rate of change** occurs when **u** has the same direction as ∇f. **Moreover, the maximum value** of the directional derivative is $|\nabla f|$.

Example 6: Let $f(x, y) = xe^y$.

a.) Find the rate of change of f at the point $(2, 0)$ in the direction of the point $P(2, 0)$ to the point $Q\left(\frac{1}{2}, 2\right)$.

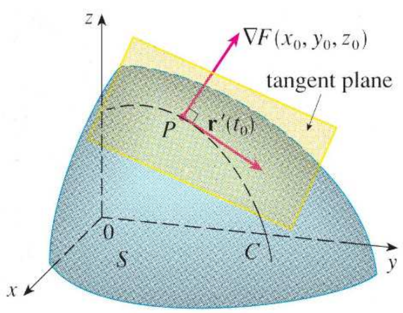
b.) At the point $(2, 0)$, in what direction does f have the maximum rate of change? What is the maximum rate of change?

Example 7: Find the maximum rate of change of $f(x, y) = \tan(3x + 2y)$ at the point $\left(\frac{\pi}{6}, -\frac{\pi}{8}\right)$ and the direction in which it occurs.

Tangent planes to level surfaces: Suppose S is a surface with equation $F(x, y, z) = k$, that is, it is a level surface of a function F, and let $P(x_0, y_0, z_0)$ be a point on S. Let C be any curve that lies on S and passes through P. Recall from section 11.6 that we can represent the curve C by the vector function $\mathbf{r(t)}) = \langle x(t), y(t), z(t)\rangle$. Let t_0 be the parameter that corresponds to the point $P(x_0, y_0, z_0)$, that is $\mathbf{r}(t_0) = \langle x_0, y_0, z_0\rangle$. Now, since C lies on the surface S, it must satisfy the equation of the surface, that is $F(x(t), y(t), z(t)) = k$. Using the chain rule and differentiating both sides with respect to t, we obtain

$\dfrac{\partial F}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial F}{\partial y}\dfrac{dy}{dt} + \dfrac{\partial F}{\partial z}\dfrac{dz}{dt} = 0$. This is equivalent to

$\left\langle \dfrac{\partial F}{\partial x}, \dfrac{\partial F}{\partial y}, \dfrac{\partial F}{\partial z} \right\rangle \cdot \left\langle \dfrac{dx}{dt}, \dfrac{dy}{dt}, \dfrac{dz}{dt} \right\rangle = 0$, thus $\nabla f \cdot \mathbf{r}'(\mathbf{t}) = 0$. Thus the gradient vector is perpendicular to the tangent vector. Hence we define the **tangent plane to the level surface** $F(x, y, z) = k$ **at the point** $P(x_0, y_0, z_0)$ **to be the plane that passes thru** $P(x_0, y_0, z_0)$ **and has normal vector** $\nabla f(x_0, y_0, z_0)$**.**



Definition:

(i) The **tangent plane** to the level surface $F(x, y, z) = k$ at the point $P(x_0, y_0, z_0)$ is

$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0.$

(ii) The **normal line** to the surface S at the point $P(x_0, y_0, z_0)$ is the line thru P perpendicular to the tangent plane, thus the normal line has direction vector ∇F.

Example 8: Find the tangent plane and normal line to the ellipsoid

$\dfrac{x^2}{4} + y^2 + \dfrac{z^2}{9} = 3$ at the point $(-2, 1, -3)$.

Example 9: Find the tangent plane to the surface $x = y^2 + z^2 + 1$ at the point $(3, 1, -1)$

Example 10: The temperature at a point (x, y, z) is given by $T(x, y, z) = 200e^{-x^2-3y^2-9z^2}$ where T is measured in degrees celsius and x, y, z in meters.

a.) Find the rate of change of temperature at the point $(2, -1, 2)$ in the direction toward the point $(3, -3, 3)$.

b.) In what direction does the temperature increase fastest at P?

c.) Find the maximum rate of change at P.