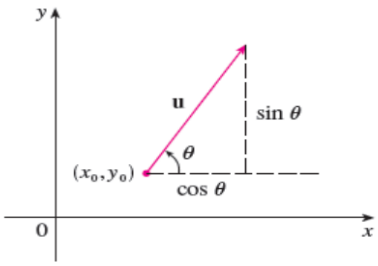
Section 14.6 Directional Derivatives

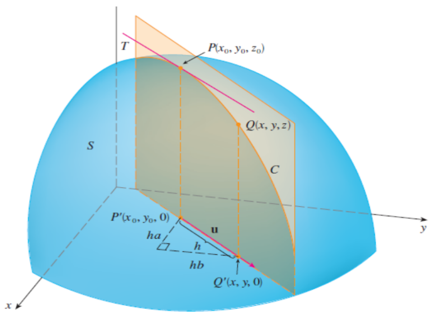
Recall: If $z = f(x, y)$, then $f_x(x_0, y_0)$ is the rate of change of z in the x-direction while y is held constant. Another way to view this is, $f_x(x_0, y_0)$ is the rate of change of z in the direction of the unit vector $\langle 1, 0\rangle$. Similarly, $f_y(x_0, y_0)$ is the rate of change of z in the direction of the unit vector $\langle 0, 1\rangle$.

Suppose now we wish to find the rate of change of z at (x_0, y_0) in the direction of an arbitrary unit vector $\mathbf{u} = \langle a, b\rangle$. To do this, we consider the surface S with equation $z = f(x, y)$ and let $z_0 = f(x_0, y_0)$. Then the point $P(x_0, y_0, z_0)$ lies on S. The vertical plane that passes thru P in the direction of $\mathbf{u}$ intersects the surface in a curve C. The slope of the tangent line T to C at P is the rate of change of z in the direction of $\mathbf{u}$, called the **directional derivative**.

Definition: The **directional derivative** of $z = f(x, y)$ at (x_0, y_0) in the direction of a unit vector $\mathbf{u} = \langle a, b\rangle$ is $D_{\mathbf{u}} f(x_0, y_0) = f_x(x_0, y_0)a + f_y(x_0, y_0)b = \langle f_x(x, y), f_y(x, y)\rangle \cdot \langle a, b\rangle$.

$z = f(x,y)$ gradient of $f(x,y)$





Example 1: Given $f(x, y) = x^3 - 4x^2y + y^2$, find the directional derivative at the point $(0, -1)$ in the direction $\mathbf{u} = \left\langle \frac{3}{5}, \frac{4}{5} \right\rangle$.

$f_x = 3x^2 - 8xy \rightarrow f_x(0,-1) = 0$

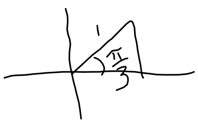
$f_y = -4x^2 + 2y \rightarrow f_y(0,-1) = -2$

$D_u f(0,-1) = \langle f_x(0,-1), f_y(0,-1)\rangle \cdot \langle \frac{3}{5}, \frac{4}{5}\rangle$

$= \langle 0, -2\rangle \cdot \langle \frac{3}{5}, \frac{4}{5}\rangle = 0 - \frac{8}{5} = \boxed{-\frac{8}{5}}$

Example 2: Suppose $f(x, y) = y^2 + 2xy$. Find $D_{\mathbf{u}} f(x, y)$ at the point $(2, 3)$ where $\mathbf{u}$ is the unit vector corresponding to $\frac{\pi}{3}$.

$\vec{u} = \langle \cos\frac{\pi}{3}, \sin\frac{\pi}{3}\rangle$



$\vec{u} = \langle \frac{1}{2}, \frac{\sqrt{3}}{2}\rangle$

$f_x = 2y$ $\qquad$ $f_y = 2y + 2x$

$f_x(2,3) = 6$ $\qquad$ $f_y(2,3) = 10$

$D_u f(2,3) = \langle 6, 10\rangle \cdot \langle \frac{1}{2}, \frac{\sqrt{3}}{2}\rangle$

$= \boxed{3 + 5\sqrt{3}}$

Example 3: Find $D_{\mathbf{u}} f(x, y)$ at the point $(1, 2)$ in the direction of $\langle 1, -3\rangle$ to the surface $f(x, y) = x^3 + 2x^2y^2$.

$\vec{u}$ must be unit!

$\vec{u} = \dfrac{\langle 1, -3 \rangle}{\sqrt{10}}$

$\vec{u} = \left\langle \dfrac{1}{\sqrt{10}}, \dfrac{-3}{\sqrt{10}} \right\rangle$

$\nabla f = \langle f_x, f_y \rangle$

$\nabla f = \langle 3x^2 + 4xy^2, 4x^2y \rangle$

$\nabla f(1, 2) = \langle 3 + 4(4), 4(2) \rangle = \langle 19, 8 \rangle$

$D_u f(1,2) = \langle 19, 8 \rangle \cdot \left\langle \dfrac{1}{\sqrt{10}}, \dfrac{-3}{\sqrt{10}} \right\rangle = \dfrac{19}{\sqrt{10}} - \dfrac{24}{\sqrt{10}}$

$= \dfrac{-5}{\sqrt{10}}$

Example 4: If $f(x, y, z) = z^3 - x^2y$, find $D_{\mathbf{u}} f(1, 6, 2)$ if $\mathbf{u} = \left\langle \dfrac{3}{13}, \dfrac{4}{13}, \dfrac{12}{13} \right\rangle$.

$\nabla f = \langle f_x, f_y, f_z \rangle = \langle -2xy, -x^2, 3z^2 \rangle$

$\nabla f(1, 6, 2) = \langle -2(6), -1, 12 \rangle = \langle -12, -1, 12 \rangle$

$D_u f(1, 6, 2) = \nabla f \cdot u = \langle -12, -1, 12 \rangle \cdot \left\langle \dfrac{3}{13}, \dfrac{4}{13}, \dfrac{12}{13} \right\rangle$

$= -\dfrac{36}{13} - \dfrac{4}{13} + \dfrac{144}{13} = \boxed{\dfrac{104}{13}}$

Definition: We define the **gradient** of $z = f(x, y)$ to be the vector $\nabla f = \langle f_x(x, y), f_y(x, y) \rangle$. Therefore,
$D_{\mathbf{u}} f(x_0, y_0) = \langle f_x(x_0, y_0), f_y(x_0, y_0) \rangle \cdot \langle a, b \rangle = \nabla f \cdot \mathbf{u}$.

Example 5: If $f(x, y) = e^x - \cos(xy)$, find $\nabla f(1, 0)$.

$\nabla f = \langle f_x, f_y \rangle$

$= \langle e^x + y \sin(xy), x \sin(xy) \rangle$

$\nabla f(1, 0) = \langle e, 0 \rangle$

Fact: If we consider all possible directional derivatives at a given point, the **maximum rate of change** occurs when **u** has the same direction as ∇f. **Moreover, the maximum value** of the directional derivative is $|\nabla f|$.

$D_u f(x,y) = \vec{\nabla f} \cdot \vec{u}$

$= |\vec{\nabla f}||\vec{u}|\cos\theta$

a maximum $= |\vec{\nabla f}|$

maximized if $\theta = 0$ since $\cos 0 = 1$

$|u| = 1$

so $\vec{u}$ is the same direction of $\vec{\nabla f}$

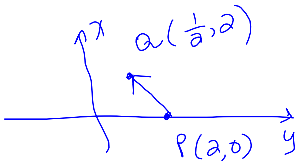
direction of the greatest rate of change is ∇f (a vector)

the maximum rate of change is $|\nabla f|$ (a number)

Example 6: Let $f(x,y) = xe^y$.

a.) Find the rate of change of f at the point $(2,0)$ in the direction of the point $P(2,0)$ to the point $Q\left(\frac{1}{2}, 2\right)$.

$D_u f(2,0)$



$\vec{u} = \frac{\vec{PQ}}{|\vec{PQ}|} = \frac{\langle -\frac{3}{2}, 2\rangle}{\sqrt{\frac{9}{4}+4}}$

$= \frac{\langle -\frac{3}{2}, 2\rangle}{\sqrt{\frac{25}{4}}}$

$\boxed{\vec{u} = \frac{2}{5}\langle -\frac{3}{2}, 2\rangle = \langle -\frac{3}{5}, \frac{4}{5}\rangle}$

$\nabla f = \langle e^y, xe^y\rangle$

$\boxed{\nabla f(2,0) = \langle 1, 2\rangle}$

$D_u f(2,0) = \nabla f \cdot u = \langle 1,2\rangle \cdot \langle -\frac{3}{5}, \frac{4}{5}\rangle$

$= -\frac{3}{5} + \frac{8}{5} = \boxed{1}$

b.) At the point $(2,0)$, in what direction does f have the maximum rate of change? What is the maximum rate of change?

① direction of maximum rate of change is $\nabla f(2,0) = \langle 1,2\rangle$

② maximum rate of change is $|\nabla f(2,0)| = \sqrt{5}$

Example 7: Find the maximum rate of change of $f(x,y) = \tan(3x+2y)$ at the point $\left(\frac{\pi}{6}, -\frac{\pi}{8}\right)$ and the direction in which it occurs.

max ROC is $|\nabla f(\frac{\pi}{6}, -\frac{\pi}{8})|$.

$3(\frac{\pi}{6}) + 2(-\frac{\pi}{8}) = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$

direction of max ROC is $\nabla f(\frac{\pi}{6}, -\frac{\pi}{8})$

$\nabla f = \langle f_x, f_y\rangle = \langle 3\sec^2(3x+2y), 2\sec^2(3x+2y)\rangle$

$= \langle 3\sec^2(\frac{\pi}{4}), 2\sec^2(\frac{\pi}{4})\rangle$

$\cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$

$\sec\frac{\pi}{4} = \frac{2}{\sqrt{2}}$

$\sec^2\frac{\pi}{4} = \frac{4}{2} = 2$

$\boxed{\nabla f = \langle 6, 4\rangle = \text{direction}}$

max ROC $= \sqrt{36+16} = \sqrt{52}$

Recall: the equation of the tangent plane to the surface $z=f(x,y)$ at the point (x_0,y_0,z_0) is

$$z-z_0=f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0)$$

$\vec{n}=\langle f_x, f_y\rangle$

Tangent planes to level surfaces: Suppose S is a surface with equation $\boxed{F(x,y,z)=k}$, that is, it is a level surface of a function F, and let $P(x_0,y_0,z_0)$ be a point on S. Let C be any curve that lies on S and passes through P. Recall from section 11.6 that we can represent the curve C by the vector function $\mathbf{r}(\mathbf{t})) = \langle x(t), y(t), z(t)\rangle$. Let t_0 be the parameter that corresponds to the point $P(x_0,y_0,z_0)$, that is $\mathbf{r}(t_0)=\langle x_0,y_0,z_0\rangle$. Now, since C lies on the surface S, it must satisfy the equation of the surface, that is $F(x(t),y(t),z(t))=k$. Using the chain rule and differentiating both sides with respect to t, we obtain

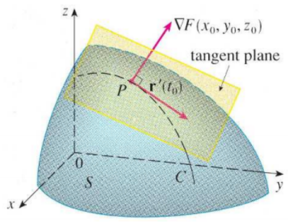
$\dfrac{\partial F}{\partial x}\dfrac{dx}{dt}+\dfrac{\partial F}{\partial y}\dfrac{dy}{dt}+\dfrac{\partial F}{\partial z}\dfrac{dz}{dt}=0.$ This is equivalent to

F

x y z

t t t

$\left\langle \dfrac{\partial F}{\partial x},\dfrac{\partial F}{\partial y},\dfrac{\partial F}{\partial z}\right\rangle\cdot\left\langle \dfrac{dx}{dt},\dfrac{dy}{dt},\dfrac{dz}{dt}\right\rangle=0$, thus $\nabla f\cdot\mathbf{r}'(\mathbf{t})=0$. Thus the gradient vector is perpendicular to the tangent vector. Hence we define the **tangent plane to the level surface** $F(x,y,z)=k$ **at the point** $P(x_0,y_0,z_0)$ **to be the plane that passes thru** $P(x_0,y_0,z_0)$ **and has normal vector** $\nabla f(x_0,y_0,z_0)$**.**



$n\cdot(r-r_0)=0$

$\nabla F\cdot\langle x-x_0, y-y_0, z-z_0\rangle=0$

Definition:

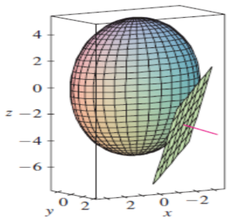
(i) The **tangent plane** to the level surface $F(x,y,z)=k$ at the point $P(x_0,y_0,z_0)$ is

$F_x(x_0,y_0,z_0)(x-x_0)+F_y(x_0,y_0,z_0)(y-y_0)+F_z(x_0,y_0,z_0)(z-z_0)=0.$

(ii) The **normal line** to the surface S at the point $P(x_0,y_0,z_0)$ is the line thru P perpendicular to the tangent plane, thus the normal line has direction vector ∇F.

Example 8: Find the tangent plane and normal line to the ellipsoid

$\dfrac{x^2}{4}+y^2+\dfrac{z^2}{9}=3$ at the point $(-2,1,-3)$.



Level surface is of the form $F(x,y,z)=k$

$F(x,y,z)=\dfrac{x^2}{4}+y^2+\dfrac{z^2}{9}$

tangent plane to level surface,

$\vec{n}=\nabla f(-2,1,-3)$

$\nabla f=\langle f_x, f_y, f_z\rangle=\left\langle \dfrac{x}{2}, 2y, \dfrac{2z}{9}\right\rangle$

$\vec{n}=\vec{\nabla} f(-2,1,-3)=\left\langle -1, 2, -\dfrac{6}{9}\right\rangle=\left\langle -1,2,-\dfrac{2}{3}\right\rangle$

$n\cdot(r-r_0)=0$

tangent plane $\boxed{\left\langle -1,2,-\dfrac{2}{3}\right\rangle\cdot\langle x+2, y-1, z+3\rangle=0}$

normal line r_0+tv gradient is direction vector

$\langle -2,1,-3\rangle+t\left\langle -1,2,-\dfrac{2}{3}\right\rangle$

Example 9: Find the tangent plane to the surface $x = y^2 + z^2 + 1$ at the point $(3, 1, -1)$

$x - y^2 - z^2 = 1$ level surface to

$F(x, y, z) = x - y^2 - z^2$

$\vec{n} = \nabla F = \langle f_x, f_y, f_z \rangle$

$= \langle 1, -2y, -2z \rangle$

$\nabla f(3,1,-1) = \langle 1, -2, 2 \rangle$

$n \cdot (r - r_0) = 0 \rightarrow \langle 1, -2, 2 \rangle \cdot \langle x-3, y-1, z+1 \rangle = 0$

Example 10: The temperature at a point (x, y, z) is given by $T(x, y, z) = 200e^{-x^2-3y^2-9z^2}$ where T is measured in degrees celsius and x, y, z in meters.

a.) Find the rate of change of temperature at the point $(2, -1, 2)$ (P) in the direction toward the point $(3, -3, 3)$ (Q).

$D_u T(2,-1,2) = \nabla f(2,-1,2) \cdot \vec{u}$

$\square = -x^2 - 3y^2 - 9z^2$

$\vec{u} = \frac{PQ}{|PQ|} = \frac{\langle 1, -2, 1 \rangle}{\sqrt{6}} = \langle \frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}} \rangle$

$\nabla T = \langle f_x, f_y, f_z \rangle = \langle -400x e^{\square}, -1200y e^{\square}, -3200z e^{\square} \rangle$

$\nabla T(2,-1,2) = \langle -800e^{-43}, 1200e^{-43}, -7200e^{-43} \rangle$

then calculate $\nabla T \cdot \vec{u} = \frac{-10400}{\sqrt{6}} e^{-43} \; \frac{{}^\circ C}{m}$

b.) In what direction does the temperature increase fastest at P?

$\nabla T(2,-1,2)$

c.) Find the maximum rate of change at P.

max ROC $= |\nabla T(2,-1,2)|$

$= \left[\sqrt{(800e^{-43})^2 + (1200e^{-43})^2 + (-7200e^{-43})^2} \right] \frac{{}^\circ C}{min}$