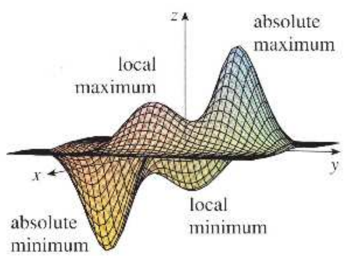
Section 14.7 Maximum and Minimum Values

Definition of local and absolute extrema:

I. A function $f(x,y)$ has a **local maximum** at (a,b) if $f(x,y) \leq f(a,b)$ for all (x,y) in a disk with center (a,b). In this case we call $f(a,b)$ a **local maximum value** of $f(x,y)$. Similarly, $f(x,y)$ has a **local minimum** at (a,b) if $f(x,y) \geq f(a,b)$ for all (x,y) in a disk with center (a,b). In this case we call $f(a,b)$ a **local minimum value** of $f(x,y)$.

II. A function $f(x,y)$ has an **absolute maximum** at (a,b) if $f(x,y) \leq f(a,b)$ for all (x,y) in the domain of f. In this case we call $f(a,b)$ the **absolute maximum value** of $f(x,y)$. Similarly, $f(x,y)$ has an **absolute minimum** at (a,b) if $f(x,y) \geq f(a,b)$ for all (x,y) in the domain of f. In this case we call $f(a,b)$ the **absolute minimum value** of $f(x,y)$.



Definition: A point (a,b) such that $f_x(a,b)=0$ and $f_y(a,b)=0$ is called a **critical point** of $f(x,y)$.

Example 1: Find all critical points of $f(x,y) = 3 - x^2 + 2x - y^2 - 4y$

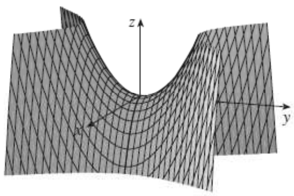
Example 2: Find all critical points of $f(x,y) = x^4 + y^4 - 4xy + 1$

The Second Derivative Test for Local Extrema: Suppose the second partial derivatives of f are continuous on a disk with center (a, b), and suppose $f_x(a, b) = 0$ and $f_y(a, b) = 0$. Let $D = D(a, b) = f_{xx}(a, b)f_{yy}(a, b) - (f_{xy}(a, b))^2$.

a.) If $D > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local minimum.

b.) If $D > 0$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a local maximum.

c.) If $D < 0$, then $f(a, b)$ is called a saddle point.



d.) If $D = 0$, the second derivative test gives no information.

Example 3: Find all local extrema or saddle points for:

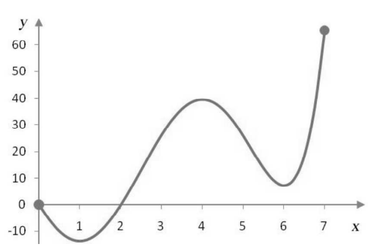
a.) $f(x, y) = y^3 - 6y^2 - 2x^3 - 6x^2 + 48x + 20$

b.) $f(x, y) = x^3 + 6xy - 2y^2$

Example 4: Suppose we have 64 square feet of cardboard in order to make a closed rectangular box with length l, width w, and height h. Find a formula for the volume of the box **in terms of** l **and** w **only** that we would use in order to maximize the volume of this box.

Example 5: A box with no lid is to hold 10 cubic meters. Find the dimensions of the box with a minimum surface area.

Recall from calculus 1 that if $y = f(x)$ is continuous on the closed interval $[a, b]$, them f will attain both an absolute maximum and an absolute minimum on the interval $[a, b]$. Moreover, the absolute extrema will occur at either an interior critical value or at one (or both) of the endpoints. This is determined by evaluating f at each interior critical value and at each endpoint. The largest of these numbers is the absolute maximum and the smallest of these numbers is the absolute minimum.



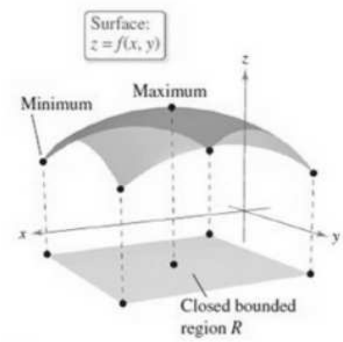
Extreme Value Theorem for Functions of Two Variables: If f is continuous on a closed, bounded set D in R^2, then f attains an absolute maximum and an absolute minimum on D.

To find the absolute extrema of a continuous function on a closed, bounded set D:

1. Find the values of f at the critical points of f in D.

2. Find the extreme values of f on the boundary of D.

3. The largest of the values from steps 1 and 2 is the absolute maximum; the smallest of these values is the absolute minimum.



Example 6: Find the absolute extrema of $f(x,y) = x^2 + xy + y^2 - 6y$ on the closed boundary

$D = \{(x,y) | -3 \le x \le 3, 0 \le y \le 5\}$.

Example 7: Find the absolute maximum of $f(x, y) = xy^2 + 8$ on the set $D = \{(x, y)\,|x^2 + y^2 \leq 3\}$.

Example 8: Find the absolute extrema of $f(x, y) = x^2 + y^2 - 2x$ on the closed triangular region with vertices $(2, 0)$, $(0, 2)$, $(0, -2)$.