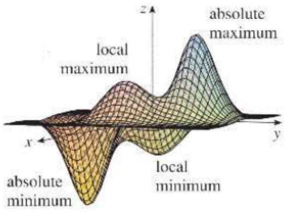
Section 14.7 Maximum and Minimum Values

Definition of local and absolute extrema:

I. A function $f(x,y)$ has a **local maximum** at (a,b) if $f(x,y) \leq f(a,b)$ for all (x,y) in a disk with center (a,b). In this case we call $f(a,b)$ a **local maximum value** of $f(x,y)$. Similarly, $f(x,y)$ has a **local minimum** at (a,b) if $f(x,y) \geq f(a,b)$ for all (x,y) in a disk with center (a,b). In this case we call $f(a,b)$ a **local minimum value** of $f(x,y)$.

II. A function $f(x,y)$ has an **absolute maximum** at (a,b) if $f(x,y) \leq f(a,b)$ for all (x,y) in the domain of f. In this case we call $f(a,b)$ the **absolute maximum value** of $f(x,y)$. Similarly, $f(x,y)$ has an **absolute minimum** at (a,b) if $f(x,y) \geq f(a,b)$ for all (x,y) in the domain of f. In this case we call $f(a,b)$ the **absolute minimum value** of $f(x,y)$.



Definition: A point (a,b) such that $f_x(a,b) = 0$ and $f_y(a,b) = 0$ is called a **critical point** of $f(x,y)$.

Example 1: Find all critical points of $f(x,y) = 3 - x^2 + 2x - y^2 - 4y$

$f_x(x,y) = -2x + 2$

$= -2(x-1)$

$\boxed{x = 1}$

$f_y(x,y) = -2y - 4$

$= -2(y+2)$

$\boxed{y = -2}$

$\boxed{CP: (1,-2)}$

Example 2: Find all critical points of $f(x,y) = x^4 + y^4 - 4xy + 1$

$f_x(x,y) = 4x^3 - 4y$

$f_y(x,y) = 4y^3 - 4x$

$f_x = 0: \boxed{4x^3 - 4y = 0}$

$y = x^3$

$f_y = 0: \quad 4y^3 - 4x = 0$

$4(x^3)^3 - 4x = 0$

$4x^9 - 4x = 0$

$4x(x^8 - 1) = 0$

$x = 0, \ x = 1, \ x = -1$

$x = 0, \ y = 0$

$x = 1, \ y = 1$

$x = -1, \ y = -1$

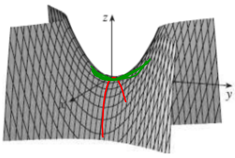
$$\boxed{\begin{array}{c}(0,0)\\(1,1)\\(-1,-1)\end{array}}$$

The Second Derivative Test for Local Extrema: Suppose the second partial derivatives of f are continuous on a disk with center (a, b), and suppose $f_x(a, b) = 0$ and $f_y(a, b) = 0$. Let $D = D(a, b) = \boxed{f_{xx}(a, b)f_{yy}(a, b) - (f_{xy}(a, b))^2}$.

a.) If $D > 0$ and $\underline{f_{xx}(a, b) > 0}$, then $f(a, b)$ is a local minimum.

b.) If $D > 0$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a local maximum.

c.) If $D < 0$, then $f(a, b)$ is called a saddle point.



d.) If $D = 0$, the second derivative test gives no information.

Example 3: Find all local extrema or saddle points for:

a.) $f(x, y) = y^3 - 6y^2 - 2x^3 - 6x^2 + 48x + 20$

local maximum value(s)		76
local minimum value(s)		-172
saddle point(s)	(x, y, f) =	(-4, 0, -140) (2, 4, 44)

① critical points

$f_x = 0$

$f_y = 0$

$\rightarrow f_x = -6x^2 - 12x + 48$

$= -6(x^2 + 2x - 8)$

$= -6(x+4)(x-2)$

$f_x = 0$ when $\boxed{x = -4,\ x = 2}$

$\rightarrow f_y = 3y^2 - 12y$

$= 3y(y-4)$

$f_y = 0$ when $\boxed{y = 0,\ y = 4}$

$$\boxed{\begin{array}{l} CP: (-4,0),\ (-4,4) \\ \qquad (2,0),\ (2,4) \end{array}}$$

②

$D = f_{xx} f_{yy} - (f_{xy})^2$

$f_{xx} = -12x - 12 = -12(x+1)$

$f_{yy} = 6y - 12 = 6(y-2)$

$f_{xy} = 0$

$D = (-12)(x+1)(6)(y-2) - 0$

$= -72(x+1)(y-2)$

CP	$D = -72(x+1)(y-2)$	$f_{xx} = -12(x+1)$	conclusion
(-4, 0)	$D = \ominus\ominus\ominus$ $D < 0$		SP (-4, 0, -140)
(-4, 4)	$D = \ominus\ominus\oplus$ $D > 0$	$f_{xx} = \ominus\ominus$ $f_{xx} > 0$	min: $z = -172$
(2, 0)	$D = \ominus\oplus\ominus$ $D > 0$	$f_{xx} = \ominus\oplus$ $f_{xx} < 0$	max: $z = 76$
(2, 4)	$D = \ominus\oplus\oplus$ $D < 0$		SP: (2, 4, 44)

b.) $f(x, y) = x^3 + 6xy - 2y^2$

local maximum value(s)		$27/2$
local minimum value(s)		DNE
saddle point(s)	$(x, y, f) =$	$(0,0,0)$

① CP $f_x = 3x^2 + 6y$

$f_y = 6x - 4y$

$3x^2 + 6y = 0$

$6x - 4y = 0 \rightarrow 6x = 4y$

$\boxed{y = \frac{6x}{4} = \frac{3}{2}x}$

$3x^2 + 6\left(\frac{3}{2}x\right) = 0$

$3x^2 + 9x = 0$

$3x(x+3) = 0$

$x = 0$
$x = -3$

$\boxed{\text{CP: } (0,0), (-3, -\tfrac{9}{2})}$

$x = 0,\ y = 0$

$x = -3,\ y = -\frac{9}{2}$

$f_{xx} = 6x,\ f_{yy} = -4$

$f_{xy} = 6$

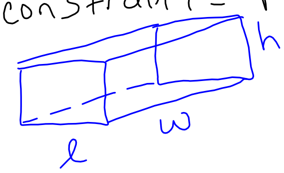
$D = f_{xx} f_{yy} - (f_{xy})^2$

$D = -24x - 36$

CP	$D = -24x - 36$	$f_{xx} = 6x$	conclusion
$(0,0)$	$D < 0$		SP $(0,0,0)$
$(-3, -\frac{9}{2})$	$D > 0$	$f_{xx} < 0$	max $z = \frac{27}{2}$

Example 4: Suppose we have 64 square feet of cardboard in order to make a closed rectangular box with length l, width w, and height h. Find a formula for the volume of the box **in terms of** l **and** w **only** that we would use in order to maximize the volume of this box.

constraint = part of problem that is fixed



$A(l, w, h) = 2lw + 2wh + 2lh$

$\boxed{64 = 2lw + 2wh + 2lh}$

solve for h

$64 - 2lw = 2wh + 2lh$

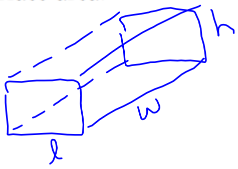
$64 - 2lw = h(2w + 2l)$

$h = \frac{64 - 2lw}{2w + 2l}$

maximize $V(l, w, h) = lwh$

$\boxed{V(l, w) = lw\left(\frac{64 - 2lw}{2w + 2l}\right)}$

Example 5: A box with no lid is to hold 10 cubic meters. Find the dimensions of the box with a minimum surface area.



constraint: $V = 10$

$\boxed{\ell w h = 10} \to h = \frac{10}{\ell w}$

minimize $A(\ell, w, h) = w\ell + 2wh + 2\ell h$

$A(\ell, w) = w\ell + 2w\left(\frac{10}{\ell w}\right) + 2\ell\left(\frac{10}{\ell w}\right)$

$A(w, \ell) = w\ell + \frac{20}{\ell} + \frac{20}{w}$

CP: $A_w = \ell - \frac{20}{w^2}$

$A_\ell = w - \frac{20}{\ell^2}$

$A_w = 0 \to \ell - \frac{20}{w^2} = 0$

$A_\ell = 0 \to w - \frac{20}{\ell^2} = 0$

$\ell - \frac{20}{w^2} = 0$

$\ell = \frac{20}{w^2}$

$w - 20\left(\frac{1}{\left(\frac{20}{w^2}\right)^2}\right) = 0$

$w - 20\left(\frac{w^4}{400}\right) = 0$

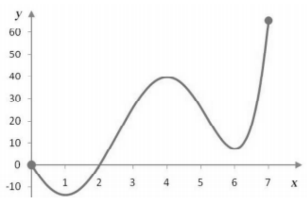
$w - \frac{w^4}{20} = 0 \to 20w - w^4 = 0$

$w(20 - w^3) = 0$

$\cancel{w = 0}$ $\boxed{w = \sqrt[3]{20}}$

solve for ℓ & h using above equations

Recall from calculus 1 that if $y = f(x)$ is continuous on the closed interval $[a, b]$, them f will attain both an absolute maximum and an absolute minimum on the interval $[a, b]$. Moreover, the absolute extrema will occur at either an interior critical value or at one (or both) of the endpoints. This is determined by evaluating f at each interior critical value and at each endpoint. The largest of these numbers is the absolute maximum and the smallest of these numbers is the absolute minimum.



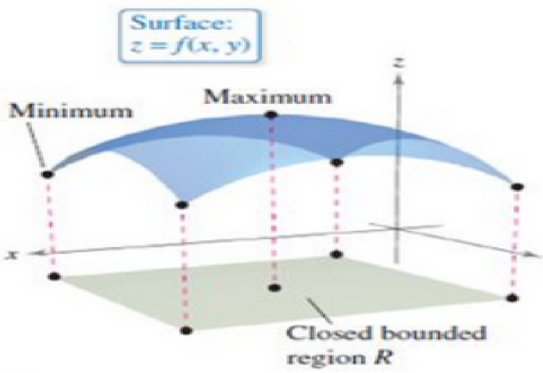
Extreme Value Theorem for Functions of Two Variables: If f is continuous on a closed, bounded set D in R^2, then f attains an absolute maximum and an absolute minimum on D.

To find the absolute extrema of a continuous function on a closed, bounded set D:

1. Find the values of f at the critical points of f in D.

2. Find the extreme values of f on the boundary of D.

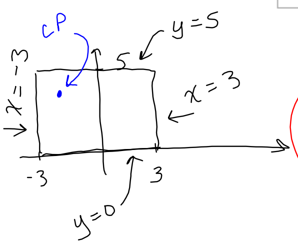
3. The largest of the values from steps 1 and 2 is the absolute maximum; the smallest of these values is the absolute minimum.



Example 6: Find the absolute extrema of $f(x,y) = x^2 + xy + y^2 - 6y$ on the closed boundary $D = \{(x,y) | -3 \le x \le 3, 0 \le y \le 5\}$.

absolute maximum value $\boxed{19}$

absolute minimum value $\boxed{-12}$



① CP: $f_x = 2x + y$ $f_x = 0$: $2x + y = ($

$y = -2x$

$f_y = x + 2y - 6$

$f_y = 0$: $x + 2y - 6 =$

$x - 4x - 6 = ($

$x = -2$

CP: $(-2, 4)$

CP: $(-2,4)$, whic is in D.

$\boxed{f(-2,4) = -12}$

② Boundary

$\boxed{y=0}$ $f(x, 0) = x^2$, $-3 \le x \le 3$

CP: $\boxed{f'(x,0) = 2x}$

CP: $\boxed{x = 0}$

$$\boxed{\begin{aligned} f(0,0) &= 0 \\ f(-3,0) &= 9 \\ f(3,0) &= 9 \end{aligned}}$$

$\boxed{x=3}$ $f(3,y) = 9 + 3y + y^2 - 6y$

$= y^2 - 3y + 9$, $\boxed{0 \le y \le 5}$

CP: $f'(3,y) = 2y - 3$ CP: $\boxed{y = \frac{3}{2}}$

$$\boxed{\begin{aligned} &y = \tfrac{3}{2}: \quad f(3, \tfrac{3}{2}) = \tfrac{9}{4} - \tfrac{9}{2} + 9 = 6.75 \\ &y = 0: \quad f(3,0) = 9 \\ &y = 5: \quad f(3,5) = 25 - 15 + 9 = 19 \end{aligned}}$$

$y = 5$: $f(x,5) = x^2 + 5x - 5$, $-3 \le x \le 3$

$f'(x,5) = 2x + 5$ CP: $x = -\frac{5}{2}$

$$\boxed{\begin{aligned} &f(-\tfrac{5}{2}, 5) = -11.25 \qquad f(3,5) = 19 \\ &f(-3,5) = -11 \end{aligned}}$$

$x = -3$: $f(-3, y) = 9 - 3y + y^2 - 6y$

$= y^2 - 9y + 9$, $0 \le y \le 5$

$f'(-3,y) = 2y - 9$ CP: $y = \frac{9}{2}$

$\boxed{f(-3,0) = 9, \quad f(-3,5) = -11, \quad f(-3, \tfrac{9}{2}) = -11.25}$

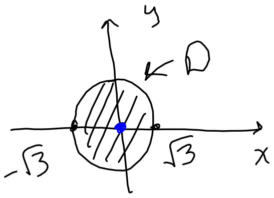
Abs max = largest of all calculated z-values which is $z = 19$

Abs min = smallest of all calculated z-values which is $z = -12$

Example 7: Find the absolute maximum of $f(x,y) = xy^2 + 8$ on the set $D = \{(x,y)\,|x^2+y^2 \leq 3\}$.

absolute maximum value: 8

absolute minimum value: 6



$f_x = y^2$ ① CP: $(0,0)$ CP is in D.

$f_y = 2xy$ $\boxed{f(0,0) = 8}$

② Boundary $x^2+y^2=3 \to y^2 = 3-x^2$

$y = \pm\sqrt{3-x^2}$

$f(x, \pm\sqrt{3-x^2}) = x(3-x^2)+8, \quad -\sqrt{3} \leq x \leq \sqrt{3}$

$= \boxed{3x - x^3 + 8}$

$f'(x, \pm\sqrt{3-x^2}) = 3 - 3x^2$ cn: $x = \pm 1$

$x = 1$: $f(1, \pm\sqrt{2}) = 3 - 1 + 8 = \boxed{10}$ Abs max

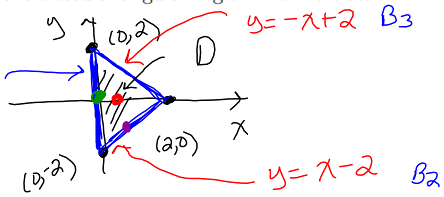
$x = -1$: $f(-1, \pm\sqrt{2}) = -3 + 1 + 8 = \boxed{6}$ Abs min

$x = \sqrt{3}$: $f(\sqrt{3}, 0) = 3\sqrt{3} - (\sqrt{3})^3 + 8 = 8$

$x = -\sqrt{3}$: $f(-\sqrt{3}, 0) = -3\sqrt{3} - (-\sqrt{3})^3 + 8 = 8$, Example 8: Find the absolute extrema of $f(x, y) = x^2 + y^2 - 2x$ on the closed triangular region with vertices $(2, 0)$, $(0, 2)$, $(0, -2)$.

absolute maximum value: 4

absolute minimum value: -1



Step1 Find all interior critical numbers that lie in D

$f_x = 0 \rightarrow 2x - 2 = 0 \rightarrow x = 1$ CP: $(1,0)$ is in D

$f_y = 0 \rightarrow 2y = 0 \rightarrow y = 0$ $f(1,0) = 1 + 0 - 2 = \boxed{-1}$

Step2: Evaluate the surface $f(x,y) = x^2 + y^2 - 2x$ on the three boundary curves.

B_1: $\boxed{x=0}$ $f(0,y) = y^2$, $\underbrace{-2 \le y \le 2}$ on interval

$f'(0,y) = 2y$ CV: $y = 0$

$y=0$: $f(0,0) = \boxed{0}$

$y=-2$: $f(0,-2) = \boxed{4}$

$y=2$: $f(0,2) = \boxed{4}$

B_2: $\boxed{y = x-2}$ $f(x, x-2) = x^2 + (x-2)^2 - 2x$, $0 \le x \le 2$

$= x^2 + x^2 - 4x + 4 - 2x$

$f(x, x-2) = \boxed{2x^2 - 6x + 4}$ $\boxed{\underbrace{0 \le x \le 2}}$

$f'(x, x-2) = 4x - 6$ CV: $x = \frac{3}{2}$ ✓

$x = \frac{3}{2}$: $f(\frac{3}{2}, -\frac{1}{2}) = 2(\frac{9}{4}) - 6(\frac{3}{2}) + 4$

$= \frac{9}{2} - \frac{18}{2} + \frac{8}{2}$

$= \boxed{-\frac{1}{2}}$

$x = 0$: $f(0,-2) = \boxed{4}$

$x = 2$: $f(2,0) = 8 - 12 + 4 = \boxed{0}$

B_3: $y = -x+2$: $f(x, -x+2) = x^2 + (-x+2)^2 - 2x$, $0 \le x \le 2$

$= \boxed{2x^2 - 6x + 4, \ 0 \le x \le 2}$

Same exact function on same exact interval, same exact output