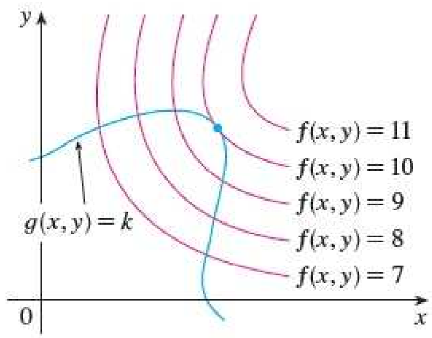
Section 14.8 Lagrange Multipliers

Lagrange Multipliers is another method used to maximize or minimize a general function $z = f(x, y)$ subject to the constraint $g(x, y) = k$. In other words, we seek the extreme values of $z = f(x, y)$ when the point (x, y) is restricted to lie on the level curve $g(x, y) = k$.

Suppose we wish to maximize $f(x, y)$ subject to the constraint $g(x, y) = k$. The figure below shows this constraint along with several level curves of the surface $f(x, y) = c$ for $c = 7, 8, 9, 10, 11$. Notice where ever the constraint level curve intersects the level curves of $f(x, y)$, we are **on the surface** $z = f(x, y)$ **subject to the constraint** $g(x, y) = k$. Every place the constraint $g(x, y)$ intersects the level curve of $f(x, y)$, we get different heights of the surface. Therefore we are looking at the point where the constraint $g(x, y)$ intersects the level curves of $f(x, y)$ **ONLY ONCE**, (otherwise the value of c could increase further and still satisfy the constrant).



In general, to maximize $f(x, y)$ subject to $g(x, y) = k$ we must find find the largest value of c such that the level curve $f(x, y) = c$ intersects $g(x, y) = k$. Likewise, to minimize $f(x, y)$ subject to $g(x, y) = k$ is to find the smallest value of c such that the level curve $f(x, y) = c$ intersects $g(x, y) = k$.

This happens when the level curves of $f(x, y)$ touches the level curve $g(x, y) = k$ **only once**, in which case both level curves have a common tangent line, and therefore therefore their normal lines are parallel. Thus their **gradient vectors are scalar multiples of each other**. Thus $\nabla f = \lambda \nabla g$ for some scalar λ. The number λ is called the **Lagrange Multiplier**.

To maximize/minimize a general function $z = f(x, y)$ **subject to a constraint of the form** $g(x, y) = k$ **(assuming that these extreme values exist)**:

1. Find all values x, y, and λ such that

   $$\nabla f(x, y) = \lambda \nabla g(x, y)$$

   and

   $$g(x, y) = k$$

2. Evaluate f at all points (x, y) that arise from the previous step. The largest of these values is the absolute maximum of f and the smallest of these values is the absolute minimum of f. Note: A similar procedure is followed for functions of three variables.

Example 1: Find the extreme values of $f(x, y) = 3x + y$ subject to the constraint $x^2 + y^2 = 10$.

Example 2: Find the extreme values of $f(x, y) = x^2 + 2y^2$ subject to the constraint $x^2 + 16y^2 = 16$.

Example 3: Find the extreme values of $f(x, y) = x^2 + y^2 + 4x - 4y$ subject to the constraint $x^2 + y^2 \leq 9$.

Example 4: Find the minimum value of $f(x, y, z) = x^2 + y^2 + z^2$ subject to the constraint $x + 3y - 2z = 12$.

Example 5: Find the volume of the largest rectangular box with faces parallel to the coordinate planes than can be inscribed in the ellipsoid $16x^2 + 4y^2 + 9z^2 = 144$.

Example 6: Find the points on the sphere $x^2 + y^2 + z^2 = 4$ that are closest to and farthest from the point $(3, 1, -1)$.