Section 14.8 Lagrange Multipliers

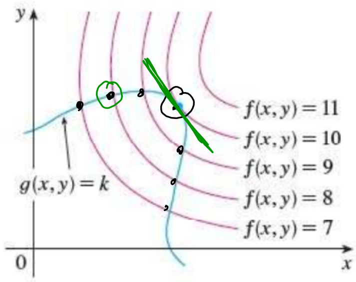
Lagrange Multipliers is another method used to maximize or minimize a general function $z = f(x, y)$ subject to the constraint $g(x, y) = k$. In other words, we seek the extreme values of $z = f(x, y)$ when the point (x, y) is restricted to lie on the level curve $g(x, y) = k$.

Suppose we wish to maximize $f(x, y)$ subject to the constraint $g(x, y) = k$. The figure below shows this constraint along with several level curves of the furface $f(x, y) = c$ for $c = 7, 8, 9, 10, 11$. Notice where ever the constraint level curve intersects the level curves of $f(x, y)$, we are **on the surface** $z = f(x, y)$ **subject to the constraint** $g(x, y) = k$. Every place the constraint $g(x, y)$ intersects the level curve of $f(x, y)$, we get different heights of the surface. Therefore we are looking at the point where the constraint $g(x, y)$ intersects the level curves of $f(x, y)$ **ONLY ONCE**, (otherwise the value of c could increase further and still satisfy the constrant).

maximize or minimize $z=f(x,y)$

subject to the constraint $g(x,y)=k$

level curve



everywhere the level curves of $f(x,y)$ intersect the constraint $g(x,y)=k$

we are on both the surface and the constraint

In general, to maximize $f(x, y)$ subject to $g(x, y) = k$ we must find find the largest value of c such that the level curve $f(x, y) = c$ intersects $g(x, y) = k$. Likewise, to minimize $f(x, y)$ subject to $g(x, y) = k$ is to find the smallest value of c such that the level curve $f(x, y) = c$ intersects $g(x, y) = k$.

This happens when the level curves of $f(x, y)$ touches the level curve $g(x, y) = k$ **only once**, in which case both level curves have a common tangent line, and therefore therefore their normal lines are parallel. Thus their **gradient vectors are scalar multiples of each other**. Thus $\nabla f = \lambda \nabla g$ for some scalar λ. The number λ is called the **Lagrange Multiplier**.

To maximize/minimize a general function $z = f(x, y)$ subject to a constraint of the form $g(x, y) = k$ (assuming that these extreme values exist):

1. Find all values x, y, and λ such that

$$\boxed{\begin{gathered}\underline{\nabla f}(x, y) = \lambda \underline{\nabla g(x, y)} \\ \text{and} \qquad g(x, y) = k\end{gathered}}$$

2. Evaluate f at all points (x, y) that arise from the previous step. The largest of these values is the absolute maximum of f and the smallest of these values is the absolute minimum of f. Note: A similar procedure is followed for functions of three variables.

Example 1: Find the extreme values of $f(x, y) = 3x + y$ subject to the constraint $x^2 + y^2 = 10$.

$g(x,y) = x^2 + y^2$

Solve $\nabla f = \lambda \nabla g$

$\langle f_x, f_y \rangle = \lambda \langle g_x, g_y \rangle$

$$\boxed{\begin{array}{l} \langle 3, 1 \rangle = \lambda \langle 2x, 2y \rangle \\ x^2 + y^2 = 10 \end{array}}$$

$3 = 2\lambda x \rightarrow x = \frac{3}{2\lambda}$

$1 = 2\lambda y \rightarrow y = \frac{1}{2\lambda}$

$x^2 + y^2 = 10$

$\frac{9}{4\lambda^2} + \frac{1}{4\lambda^2} = 10$

$\frac{10}{4\lambda^2} = 10 \rightarrow 4\lambda^2 = 1$

$\lambda = \pm \frac{1}{2}$

$\lambda = \frac{1}{2} \rightarrow x = \frac{3}{2(\frac{1}{2})} = 3$

$y = 1$

$\lambda = -\frac{1}{2} \rightarrow x = -3,\ y = -1$

$(3, 1)$

$(-3, -1)$

plug these into

$f(x,y) = 3x + y$

$f(3,1) = 9 + 1 = 10$

$f(-3,-1) = -9 - 1 = -10$

Abs max = 10

Abs min = -10

Example 2: Find the extreme values of $f(x, y) = x^2 + 2y^2$ subject to the constraint $x^2 + 16y^2 = 16$.

$\nabla f = \lambda \nabla g \longrightarrow \langle 2x, 4y \rangle = \lambda \langle 2x, 32y \rangle$ $\quad g(x,y) = x^2 + 16y^2$

$x^2 + 16y^2 = 16$

$$\boxed{\begin{array}{l} 2x = 2\lambda x \\ 4y = 32\lambda y \\ x^2 + 16y^2 = 16 \end{array}}$$

$0 = 2\lambda x - 2x$

$0 = 2x(\lambda - 1)$

$\boxed{x = 0}$, $\boxed{\lambda = 1}$

$16y^2 = 16$

$y = \pm 1$

$4y = 32y$

$\boxed{y = 0}$

$x^2 = 16$

$x = \pm 4$

Consideration points:

$(0,1), (0,-1)$

$(\pm 4, 0)$

Now plug all consideration points into $f(x,y) = x^2 + 2y^2$

$f(0, \pm 1) = 2$

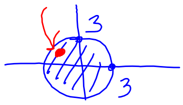
$f(\pm 4, 0) = 16$

Abs max = 16

Abs min = 2

Example 3: Find the extreme values of $f(x, y) = x^2 + y^2 + 4x - 4y$ subject to the constraint $x^2 + y^2 \leq 9$.

① if we are inside the circle $x^2+y^2=9$, we must consider all CP within the circle.



$f_x = 2x+4$ CP: $(-2,2)$

$f_y = 2y-4$ $f(-2,2) = 4+4-8-8=-8$

② on circle $x^2+y^2=9$ constraint! use Lagrange.

$g(x,y) = x^2+y^2$

$\nabla f = \lambda \nabla g$

$\langle 2x+4, 2y-4\rangle = \lambda\langle 2x, 2y\rangle$

$$\boxed{\begin{aligned} 2x+4 &= 2\lambda x \\ 2y-4 &= 2\lambda y \\ x^2+y^2 &= 9 \end{aligned}}$$

$4 = 2\lambda x - 2x$

$4 = 2x(\lambda-1)$

$\boxed{\dfrac{2}{\lambda-1} = x}$

$2y-4 = 2\lambda y$

$-4 = 2\lambda y - 2y$

$-4 = 2y(\lambda-1)$

$\boxed{y = \dfrac{-2}{\lambda-1}}$

plug $x = \dfrac{2}{\lambda-1}$, $y = \dfrac{-2}{\lambda-1}$ into $x^2+y^2=9$

$\dfrac{4}{(\lambda-1)^2} + \dfrac{4}{(\lambda-1)^2} = 9$

$8 = 9(\lambda-1)^2$

$(\lambda-1)^2 = \dfrac{8}{9}$

$\lambda - 1 = \pm\dfrac{2\sqrt{2}}{3}$

For $\lambda-1 = \dfrac{2\sqrt{2}}{3}$: $x = \dfrac{2}{\lambda-1} = \dfrac{2}{\frac{2\sqrt{2}}{3}} = \dfrac{3}{\sqrt{2}}$, $y = -\dfrac{2}{\lambda-1} = -\dfrac{3}{\sqrt{2}}$

gives consideration point of $\left(\dfrac{3}{\sqrt{2}}, -\dfrac{3}{\sqrt{2}}\right)$

For $\lambda-1 = -\dfrac{2\sqrt{2}}{3}$: $x = \dfrac{2}{\lambda-1} = \dfrac{2}{-\frac{2\sqrt{2}}{3}} = -\dfrac{3}{\sqrt{2}}$, $y = \dfrac{-2}{\lambda-1} = \dfrac{3}{\sqrt{2}}$

gives consideration point of $\left(-\dfrac{3}{\sqrt{2}}, \dfrac{3}{\sqrt{2}}\right)$

$f\left(\frac{3}{\sqrt{2}}, \frac{-3}{\sqrt{2}}\right) = 9 + \frac{24}{\sqrt{2}}$ max

$f\left(-\frac{3}{\sqrt{2}}, \frac{3}{\sqrt{2}}\right) = 9 - \frac{24}{\sqrt{2}}$ min

Example 4: Find the minimum value of $f(x, y, z) = x^2 + y^2 + z^2$ subject to the constraint $x + 3y - 2z = 12$.

$\nabla f = \lambda \nabla g$ $g(x,y,z) = x+3y-2z$

$\langle 2x, 2y, 2z\rangle = \lambda\langle 1, 3, -2\rangle$

$2x = \lambda$

$2y = 3\lambda$

$2z = -2\lambda$

$$\boxed{\begin{aligned} x &= \tfrac{1}{2}\lambda \\ y &= \tfrac{3}{2}\lambda \\ z &= -\lambda \end{aligned}}$$

$x+3y-2z = 12$

$\underbrace{\frac{1}{2}\lambda + \frac{9}{2}\lambda}_{5\lambda} + 2\lambda = 12$

$7\lambda = 12$ $\lambda = \frac{12}{7}$

$x = \frac{1}{2}\left(\frac{12}{7}\right) = \frac{6}{7}$

$y = \frac{3}{2}\left(\frac{12}{7}\right) = \frac{18}{7}$

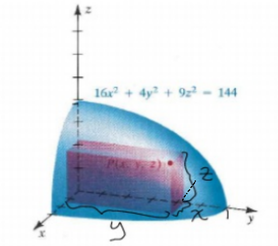
$z = -\frac{12}{7} = -\frac{12}{7}$

$f\left(\frac{6}{7}, \frac{18}{7}, -\frac{12}{7}\right) = \left(\frac{6}{7}\right)^2 + \left(\frac{18}{7}\right)^2 + \left(\frac{-12}{7}\right)^2$

$\boxed{\text{min} = \frac{72}{7}}$

Example 5: Find the volume of the largest rectangular box with faces parallel to the coordinate planes than can be inscribed in the ellipsoid $16x^2 + 4y^2 + 9z^2 = 144$.

constraint



maximize $V(x,y,z) = (2x)(2y)(2z)$

subject to $16x^2 + 4y^2 + 9z^2 = 144$

$V(x,y,z) = 8xyz$

$\nabla V = \lambda \nabla g$

$\langle 8yz, 8xz, 8xy \rangle = \lambda \langle 32x, 8y, 18z \rangle$

multiply by x → $8yz = 32\lambda x$ — ① $8xyz = 32\lambda x^2$

multiply by y → $8xz = 8\lambda y$ — ② $8xyz = 8\lambda y^2$

multiply by z → $8xy = 18\lambda z$ — ③ $8xyz = 18\lambda z^2$

$16x^2 + 4y^2 + 9z^2 = 144$

note: we can divide by λ since $\lambda \neq 0$

set ①=② and ①=③

$32\lambda x^2 = 8\lambda y^2 \rightarrow \boxed{4x^2 = y^2}$

$32\lambda x^2 = 18\lambda z^2 \rightarrow \frac{32}{18}x^2 = z^2$ or $\boxed{\frac{16}{9}x^2 = z^2}$

$16x^2 + 16x^2 + 16x^2 = 144$

$48x^2 = 144$

$x^2 = \frac{144}{48}$ so $\boxed{x = \frac{12}{\sqrt{48}}}$

note: positive root only since x represents length

plug into constraint

set ②=① and ②=③

$8\lambda y^2 = 32\lambda x^2 \rightarrow x^2 = \frac{y^2}{4}$

$8\lambda y^2 = 18\lambda z^2 \rightarrow z^2 = \frac{8}{18}y^2$ or $z^2 = \frac{4}{9}y^2$

$16\left(\frac{y^2}{4}\right) + 4y^2 + 9\left(\frac{4}{9}y^2\right) = 144$

$4y^2 + 4y^2 + 4y^2 = 144 \rightarrow 12y^2 = 144$ $\boxed{y = \sqrt{12}}$

Finally, set ③=① and ③=②

$18\lambda z^2 = 32\lambda x^2 \rightarrow x^2 = \frac{18z^2}{32} = \frac{9z^2}{16}$

$18\lambda z^2 = 8\lambda y^2$ — $y^2 = \frac{18z^2}{8} = \frac{9z^2}{4}$

plug into constraint

$16\left(\frac{9z^2}{16}\right) + 4\left(\frac{9z^2}{4}\right) + 9z^2 = 144$

$9z^2 + 9z^2 + 9z^2 = 144$

$27z^2 = 144$

$z^2 = \frac{144}{27}$, $z = \frac{12}{\sqrt{27}} = \frac{12}{3\sqrt{3}} = \frac{4}{\sqrt{3}}$

$x = \frac{12}{\sqrt{48}}$, $y = \sqrt{12}$, $z = \frac{4}{\sqrt{3}}$

$f\left(\frac{12}{\sqrt{48}}, \sqrt{12}, \frac{4}{\sqrt{3}}\right) =$ ________________

Example 6: Find the points on the sphere $x^2 + y^2 + z^2 = 4$ that are closest to and farthest from the point $(3, 1, -1)$.