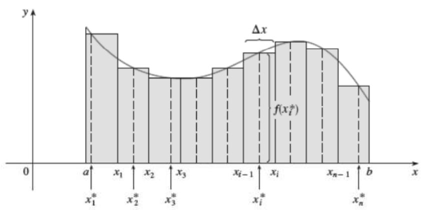
Section 15.1 Double Integrals over Rectangles

Recall in calculus 1, in order to define the definite integral of $y = f(x)$ over the interval $[a, b]$, we first took a partition of the interval $[a, b]$ into n subintervals, and for each subinterval $[x_{i-1}, x_i]$, we defined $\Delta x_i = x_i - x_{i-1}$ and x_i^* to be any point on the subinterval. We then defined the **definite integral** of $y = f(x)$ over the interval $[a, b]$ to be $\displaystyle\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x_i$.

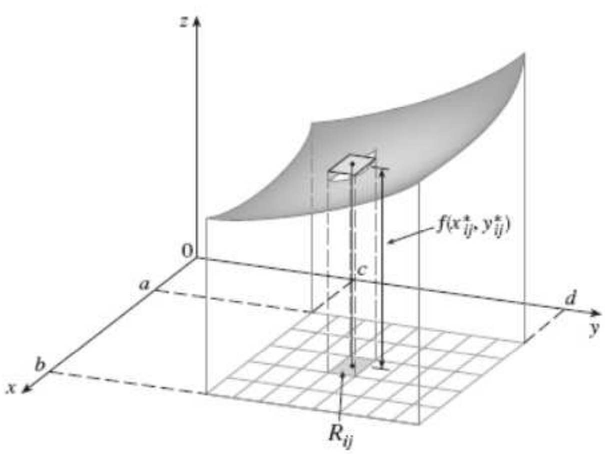


In a similar manner, we now consider a function f of two variables defined on a closed rectangle

$R = \{(x, y) | a \le x \le b, y \le c \le d\}$. We take a partition of R into sub-rectangles, and as before,

$\Delta x_i = x_i - x_{i-1}$ and $\Delta y_i = y_j - y_{j-1}$, and (x_{ij}^*, y_{ij}^*) is any point in the subrectangle R_{ij}. If the area of R_{ij} is $\Delta A_{ij} = \Delta x_i \Delta y_j$, then we define the **Double Integral** of $f(x, y)$ over the rectangle R to be

$$\iint_R f(x, y)\,dA = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f((x_{ij}^*, y_{ij}^*))\Delta A_{ij}$$



Iterated Integrals

Suppose $z = f(x, y)$ is a function of two variables that is integrable over the rectangle $R = [a, b] \times [c, d]$.

a.) We use the notation $\int_a^b f(x, y)\, dx$ to mean that y is held fixed and $f(x, y)$ is integrated with respect to x from $x = a$ to $x = b$. This is called **partial integration with respect to** x.

b.) We use the notation $\int_c^d f(x, y)\, dy$ to mean that x is held fixed and $f(x, y)$ is integrated with respect to y from $y = c$ to $y = d$. This is called **partial integration with respect to** y.

Example 1: Find $\int_2^3 (x^2 + y^3)\, dx$

Example 2: Find $\int_0^{\pi/4} x\cos(3y)\, dy$

Definition: An **iterated integral** is an integral of the form $\int_c^d \int_a^b f(x, y)\, dxdy$ or $\int_a^b \int_c^d f(x, y)\, dydx$

(i) $\int_c^d \int_a^b f(x, y)\, dxdy = \int_c^d \left[\int_a^b f(x, y)\, dx\right] dy$

(ii) $\int_a^b \int_c^d f(x, y)\, dydx = \int_a^b \left[\int_c^d f(x, y)\, dy\right] dx$

Example 3: Evaluate $\int_0^3 \int_1^2 x^2 y\, dydx$ and $\int_1^2 \int_0^3 x^2 y\, dxdy$.

Fubini's Theorem: If f is continuous on the rectangle $R = [a, b] \times [c, d]$, then

$$\iint_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dydx = \int_c^d \int_a^b f(x,y)\,dxdy.$$

Special case of Fubini: In the case where $f(x,y) = g(x)h(y)$, then

$$\iint_R f(x,y)\,dA = \int_a^b \int_c^d g(x)h(y)\,dydx = \int_a^b g(x)\,dx \int_c^d h(y)\,dy$$

Note: You will find this special case VERY useful!! Get used to recognizing when it can be applied!

Example 4: Evaluate $\displaystyle\int_1^2 \int_0^3 6x^2 y\,dxdy$

Example 5: Evaluate $\displaystyle\int_4^5 \int_0^1 \frac{1+x}{1+y}\,dxdy$

Example 6: Evaluate $\iint_R x\sqrt{1+x^2}\,dA$, where $R = \{(x, y)|0 \leq x \leq 1, 1 \leq y \leq 2\}$.

Example 7: Evaluate $\iint_R 2xy\cos(x^2y)\,dA$, where $R = \left\{(x, y)|0 \leq x \leq \sqrt{\frac{\pi}{4}}, 0 \leq y \leq 1\right\}$

Example 8: Evaluate $\displaystyle\iint_R e^{2x-y}\,dA$, where $R = [0, \ln 2] \times [0, \ln 5]$.

Example 9: Evaluate $\displaystyle\iint_R (y\sin(xy))\,dA$, where $R = [0, 2] \times [0, \pi]$.

Theorem: If $f(x, y) \geq 0$ and f is continuous on the rectangle R, then the volume V of the solid that lies above R and under the surface $f(x, y)$ is

$$V = \iint_R f(x, y)\, dA$$

Example 10: Evaluate the double integral and identify it as the volume of a solid.

$\iint_R (3 - x)\, dA$, $R = \{(x, y) | 0 \leq x \leq 3, 0 \leq y \leq 8\}$

Example 11: Find the volume of the solid S that is bounded by the paraboloid $x^2 + 2y^2 + z = 16$, the planes $x = 2$ and $y = 2$ and the three coordinate planes.