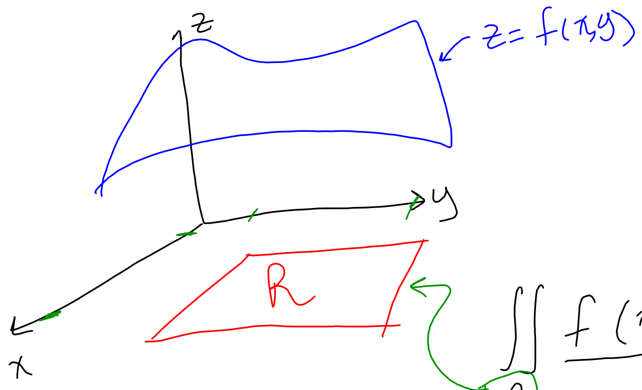
Recall:



$a \le x \le b$

$c \le y \le d$

$$\iint_R f(x,y)\,dA$$

$\hookrightarrow dx\,dy$ or $dy\,dx$

$$\int_c^d \int_a^b f(x,y)\,dx\,dy$$

$$\int_a^b \int_c^d f(x,y)\,dy\,dx$$

$$\int_a^b \int_c^d f(x,y)\,dx\,dy$$

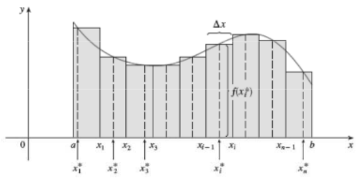
$$= \int_c^d \int_a^b f(x,y)\,dx\,dy$$

Fubini:

$$\int_a^b \int_c^d h(x)\,g(y)\,dy\,dx = \left(\int_a^b h(x)\,dx\right)\left(\int_c^d g(y)\,dy\right)$$

Section 15.1 Double Integrals over Rectangles

Recall in calculus 1, in order to define the definite integral of $y = f(x)$ over the interval $[a, b]$, we first took a partition of the interval $[a, b]$ into n subintervals, and for each subinterval $[x_{i-1}, x_i]$, we defined $\Delta x_i = x_i - x_{i-1}$ and x_i^* to be any point on the subinterval. We then defined the **definite integral** of $y = f(x)$ over the interval $[a, b]$ to be $\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x_i$.



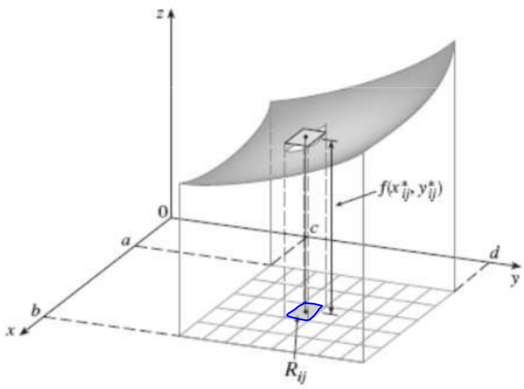
In a similar manner, we now consider a function f of two variables defined on a closed rectangle

$R = \{(x, y) | a \le x \le b, y \le c \le d\}$. We take a partition of R into sub-rectangles, and as before,

$\Delta x_i = x_i - x_{i-1}$ and $\Delta y_i = y_j - y_{j-1}$, and (x_{ij}^*, y_{ij}^*) is any point in the subrectangle R_{ij}. If the area of R_{ij} is $\Delta A_{ij} = \Delta x_i \Delta y_j$, then we define the **Double Integral** of $f(x, y)$ over the rectangle R to be

$$\iint_R f(x, y)\,dA = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f((x_{ij}^*, y_{ij}^*))\Delta A_{ij}$$

$\hookrightarrow (\Delta x_{ij})(\Delta y_{ij})$



Iterated Integrals

Suppose $z = f(x, y)$ is a function of two variables that is integrable over the rectangle $R = [a, b] \times [c, d]$.

a.) We use the notation $\int_a^b f(x, y)\, dx$ to mean that y is held fixed and $f(x, y)$ is integrated with respect to x from $x = a$ to $x = b$. This is called **partial integration with respect to** x.

b.) We use the notation $\int_c^d f(x, y)\, dy$ to mean that x is held fixed and $f(x, y)$ is integrated with respect to y from $y = c$ to $y = d$. This is called **partial integration with respect to** y.

Example 1: Find $\int_2^3 (x^2 + y^3)\, dx$ $= \left(\frac{x^3}{3} + y^3 x\right)\Big|_{x=2}^{x=3}$

$$= 9 + y^3(3) - \left(\frac{8}{3} + y^3(2)\right)$$

$$= 9 + y^3 - \frac{8}{3} = \boxed{\frac{19}{3} + y^3}$$

Example 2: Find $\int_0^{\pi/4} x\cos(3y)\, dy$ $= x\,\frac{\sin(3y)}{3}\Big|_{y=0}^{y=\frac{\pi}{4}}$

$$= \frac{x}{3}\left(\sin\frac{3\pi}{4} - \sin(0)\right)$$

$$= \boxed{\frac{x}{3}\left(\frac{\sqrt{2}}{2}\right)}$$

Definition: An **iterated integral** is an integral of the form $\int_c^d \int_a^b f(x, y)\, dxdy$ or $\int_a^b \int_c^d f(x, y)\, dydx$

(i) $\int_c^d \int_a^b f(x, y)\, dxdy = \int_c^d \left[\int_a^b f(x, y)\, dx\right] dy$

(ii) $\int_a^b \int_c^d f(x, y)\, dydx = \int_a^b \left[\int_c^d f(x, y)\, dy\right] dx$

Example 3: Evaluate ① $\int_0^3 \int_1^2 x^2 y\, dydx$ and ② $\int_1^2 \int_0^3 x^2 y\, dxdy$.

① $\int_0^3 \left[\int_1^2 x^2 y\, dy\right] dx = \int_0^3 \left[x^2 \frac{y^2}{2}\Big|_{y=1}^{y=2}\right] dx$

$$= \int_0^3 \left[\frac{x^2}{2}(4-1)\right] dx$$

$$= \int_0^3 \frac{3x^2}{2}\, dx$$

$$= \frac{\cancel{3}}{2}\,\frac{x^3}{\cancel{3}}\Big|_0^3$$

$$= \boxed{\frac{1}{2}(27)}$$

② $\int_1^2 \left[\int_0^3 x^2 y\, dx\right] dy$

$$\int_1^2 \left[\frac{x^3}{3} y \Big|_0^3\right] dy$$

$$\int_1^2 9y\, dy$$

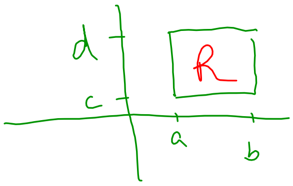
$$\frac{9y^2}{2}\Big|_1^2 = \frac{9}{2}(4-1)$$

$$= \boxed{\frac{27}{2}}$$

Same!

Fubini's Theorem: If f is continuous on the rectangle $R = [a,b] \times [c,d]$, then

$$\iint_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dydx = \int_c^d \int_a^b f(x,y)\,dxdy.$$



Special case of Fubini: In the case where $f(x,y) = g(x)h(y)$, then

$$\longrightarrow \iint_R f(x,y)\,dA = \int_a^b \int_c^d \underbrace{g(x)h(y)}\,dydx = \int_a^b g(x)\,dx \int_c^d h(y)\,dy$$

Note: You will find this special case VERY useful!! Get used to recognizing when it can be applied!

Example 4: Evaluate $\int_1^2 \int_0^3 6x^2 y\,dxdy$ $= \left(\int_0^3 6x^2\,dx\right)\left(\int_1^2 y\,dy\right)$

$$\boxed{\begin{matrix} 0 \le x \le 3 \\ 1 \le y \le 2 \end{matrix}}$$

$$\left(2x^3 \Big|_0^3\right)\left(\frac{y^2}{2}\Big|_1^2\right)$$

$$(54)\left(\frac{1}{2}(3)\right)$$

Example 5: Evaluate $\int_4^5 \int_0^1 \frac{1+x}{1+y}\,dxdy$

$4 \le y \le 5$

$0 \le x \le 1$

$$\frac{1+x}{1+y} = (1+x)\left(\frac{1}{1+y}\right)$$

Fubini!

$$\left(\int_0^1 (1+x)\,dx\right)\left(\int_4^5 \frac{1}{1+y}\,dy\right)$$

$$\left(x + \frac{x^2}{2}\right)\Big|_0^1 \left(\ln(1+y)\right)\Big|_4^5$$

$$\left(1+\frac{1}{2}\right)\left(\ln(6) - \ln(5)\right)$$

$$\boxed{\frac{3}{2}\ln\left(\frac{6}{5}\right)}$$

Example 6: Evaluate $\iint_R x\sqrt{1+x^2}\,dA$ where $R = \{(x,y) | 0 \le x \le 1, 1 \le y \le 2\}$.

$dA \to dx\,dy$ or $dy\,dx$

$$\int_0^1 \int_1^2 x\sqrt{1+x^2}\,dy\,dx = \left(\int_0^1 x\sqrt{1+x^2}\,dx\right)\int_1^2 dy$$

u-sub

$u = 1+x^2$

$du = 2x\,dx$

$$\left(\int_1^2 \frac{1}{2}\sqrt{u}\,du\right)\int_1^2 dy$$

$$\left(\frac{1}{2}\frac{2}{3}u^{\frac{3}{2}}\Big|_1^2\right)\left(y\Big|_1^2\right)$$

$$\frac{1}{3}\left(2\sqrt{2}-1\right)\left(2-1\right)$$

$$\boxed{\frac{1}{3}\left(2\sqrt{2}-1\right)}$$

Example 7: Evaluate $\iint_R 2xy\cos(x^2y)\,dA$, where $R = \left\{(x,y) | 0 \le x \le \sqrt{\frac{\pi}{4}}, 0 \le y \le 1\right\}$

Fubini does not work. cannot write the integrand as $h(x)g(y)$.

Two choices: $\begin{cases} dx\,dy \\ dy\,dx \end{cases}$

$$\int_0^{\sqrt{\frac{\pi}{4}}} \int_0^1 2xy\cos(x^2y)\,dy\,dx$$

or

$$\checkmark \int_0^1 \int_0^{\sqrt{\frac{\pi}{4}}} 2xy\cos(x^2y)\,dx\,dy$$

$$\int_0^1 \left[\int_0^{\sqrt{\frac{\pi}{4}}} \underbrace{\boxed{2xy}\cos(x^2y)\boxed{dx}}_{du}\right] dy$$

u-sub $\quad u = x^2y \begin{cases} x = \sqrt{\frac{\pi}{4}},\ u = \frac{\pi}{4}y \\ x = 0,\ u = 0 \end{cases}$

$$\frac{du}{dx} = 2xy \longrightarrow du = 2xy\,dx$$

$$\int_0^1 \left[\int_0^{\frac{\pi}{4}y} \cos u\,du\right] dy$$

$$\int_0^1 \left[\sin u \Big|_{u=0}^{u=\frac{\pi}{4}y}\right]$$

$$\int_0^1 \sin\frac{\pi}{4}y = -\frac{4}{\pi}\cos\frac{\pi}{4}y\Big|_0^1$$

$$= -\frac{4}{\pi}\left[\cos\frac{\pi}{4} - \cos(0)\right]$$

$$= \boxed{-\frac{4}{\pi}\left(\frac{\sqrt{2}}{2}-1\right)}$$

Example 8: Evaluate $\iint_R e^{2x-y}\,dA$, where $R = [0, \ln 2] \times [0, \ln 5]$.

$e^{2x-y} = e^{2x} \cdot e^{-y}$

$0 \le x \le \ln 2, \quad 0 \le y \le \ln 5$

$$\iint_R e^{2x-y}\,dA = \iint_R e^{2x} \cdot e^{-y}\,dA$$

$$= \left(\int_0^{\ln 2} e^{2x}\,dx\right)\int_0^{\ln 5} e^{-y}\,dy$$

$$\boxed{e^{a\ln b} = e^{\ln b^a}}$$

$$\boxed{e^{\ln x} = x}$$

$$= \left(\frac{1}{2}e^{2x}\Big|_0^{\ln 2}\right)\left(-e^{-y}\Big|_0^{\ln 5}\right)$$

$$= \frac{1}{2}\left[e^{2\ln 2} - 1\right]\left[-e^{-\ln 5} + 1\right]$$

$$= \frac{1}{2}\left[e^{\ln 4} - 1\right]\left[-e^{\ln \frac{1}{5}} + 1\right]$$

$$= \frac{1}{2}\left[4 - 1\right]\left[-\frac{1}{5} + 1\right] = \boxed{\frac{1}{2}(3)\left(\frac{4}{5}\right)}$$

Example 9: Evaluate $\iint_R (y\sin(xy))\,dA$, where $R = \underbrace{[0, 2]}_{x} \times \underbrace{[0, \pi]}_{y}$.

$$\underbrace{\iint_R y\sin(xy)\,dy\,dx}_{\text{parts!!}} \quad \text{or} \quad \underbrace{\iint_R y\sin(xy)\,dx\,dy}_{\text{better!}}$$

$$\int_0^{\pi}\left[\int_0^2 \boxed{y}\sin(xy)\boxed{dx}\right]dy$$

du

$u = xy$ — $x = 2,\ u = 2y$; $x = 0,\ u = 0$

$\frac{du}{dx} = y$

$du = y\,dx$

$$\int_0^{\pi}\left[\int_0^{2y} \sin(u)\,du\right]dy$$

$$\int_0^{\pi}\left[-\cos u\Big|_{u=0}^{u=2y}\right]dy$$

$$\int_0^{\pi}\left[-\cos(2y) + 1\right]dy$$

$$\left(-\frac{1}{2}\sin(2y) + y\right)\Big|_0^{\pi}$$

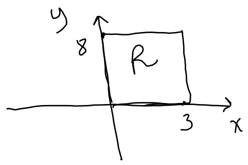
$$\boxed{\pi}$$

Theorem: If $f(x,y) \geq 0$ and f is continuous on the rectangle R, then the volume V of the solid that lies above R and under the surface $f(x,y)$ is

$$V = \iint_R f(x,y)\,dA$$

Example 10: Evaluate the double integral and identify it as the volume of a solid.

$\iint_R (3-x)\,dA$, $R = \{(x,y) | 0 \leq x \leq 3, 0 \leq y \leq 8\}$



if $\begin{cases} 0 \leq x \leq 3, \\ 0 \leq y \leq 8, \end{cases}$ $3 - x \geq 0$

$$V = \iint_R (3-x)\,dA = \int_0^3 \int_0^8 (3-x)\,dy\,dx$$

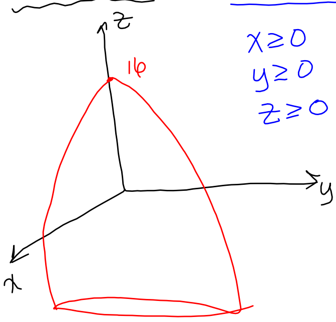
Fubini!

$$= \int_0^3 (3-x)\,dx \int_0^8 dy$$

$$= \left(3x - \frac{x^2}{2}\right)\Big|_0^3 \; y\Big|_0^8$$

$$V = \left(9 - \frac{9}{2}\right)(8)$$

Example 11: Find the volume of the solid S that is bounded by the paraboloid $x^2 + 2y^2 + z = 16$, the planes $x = 2$ and $y = 2$ and the three coordinate planes.



$x \geq 0$
$y \geq 0$
$z \geq 0$

$\boxed{\begin{matrix} 0 \leq x \leq 2 \\ 0 \leq y \leq 2 \end{matrix}}$

$z = f(x,y)$

$z = 16 - 2y^2 - x^2$

$$V = \iint_R (16 - 2y^2 - x^2)\,dA$$

$\hookrightarrow dy\,dx$

$$\int_0^2 \left[\int_0^2 (16 - 2y^2 - x^2)\,dy\right] dx$$

$$\int_0^2 \left[16y - \frac{2}{3}y^3 - x^2 y\right]\Big|_{y=0}^{y=2} dx$$

$$\int_0^2 \left[32 - \frac{16}{3} - 2x^2\right] dx$$