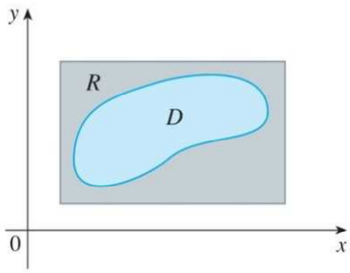
Section 15.2 Double Integrals over General Regions

For single integrals, the region over which we integrate is always an interval $[a, b]$. For double integrals, we want to be able to integrate not just over rectangles $[a, b] \times [c, d]$ but also over regions D of a more general shape. Suppose D is a bounded region, which means that D can be enclosed in a rectangular region.



There are two types of plane regions:

Type I: A plane region D is said to be of type I if it lies between the graphs of two continuous functions of x, that is $D = \{(x, y)|a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\}$.

If f is continuous on a type I region $D = \{(x, y)|a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\}$, then

$$\iint_R f(x, y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dydx$$

Type II: A plane region D is said to be of type II if it lies between the graphs of two continuous functions of y, that is $D = \{(x, y)|c \leq y \leq d, h_1(y) \leq x \leq h_2(y)\}$.

If f is continuous on a type II region $D = \{(x, y)|c \leq y \leq d, h_1(y) \leq x \leq h_2(y)\}$, then

$$\iint_D f(x, y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dxdy$$

Example 1: Sketch the region of integration and evaluate $\iint_D x\cos y\, dA$ where D is the region bounded by $y = 0$, $y = x^2$ and $x = 2$.

Example 2: Sketch the region of integration and evaluate $\int_0^2 \int_{\sqrt{x}}^3 (x^2 + y)\, dydx$.

Example 3: Sketch the region of integration and evaluate $\displaystyle\int_0^1 \int_{x-1}^0 \frac{2y}{x+1}\, dydx$.

Example 4: Sketch the region of integration and evaluate $\displaystyle\iint_D (x^2 - 2xy)\, dA$, where $D = \{(x, y)|0 \le x \le 1, \sqrt{x} \le y \le 2 - x\}$.

Example 5: Evaluate $\iint_D (x^2 + y^2)\, dA$, where

$D = \{(x, y) | 0 \leq x \leq 2, x^2 \leq y \leq 2x\}$ as both a type I and type II region.

Example 6: Set up but do not evaluate a type II integral for $\iint_D (y^2 - x)\, dA$, where D is the region bounded by $x = y^2$ and $x = 3 - 2y^2$.

Example 7: Set up but do not evaluate both a type I and type II integral for $\iint_D ye^x \, dA$, where D is the triangular region with vertices $(0,0)$, $(2,4)$ and $(6,0)$.

Example 8: Sketch the region of integration and change the order of integration.

(i) $\int_0^{\pi/2} \int_0^{\sin x} f(x,y) \, dydx$

(ii) $\int_0^1 \int_{y^2}^{2-y} f(x,y) \, dxdy$

Example 9: Evaluate $\int_0^1 \int_x^1 \sin(y^2)\, dy dx$

Theorem: If $f(x, y) \geq 0$ and f is continuous on the region R, then the volume V of the solid that lies above R and under the surface $f(x, y)$ is

$$V = \iint_R f(x, y)\, dA$$

Example 10: Find the volume of the solid under the paraboloid $z = x^2 + y^2$ above the region bounded by $y = x^2$ and $x = y^2$.

Example 11: Find the volume of the solid under the surface $z = xy$ and above the triangle with vertices $(1, 1)$, $(4, 1)$ and $(1, 2)$.

NOTE: If we integrate the constant function $f(x, y) = 1$ over a region D, we get the area of D, that is

$$\iint_D 1\,dA = A(D)$$