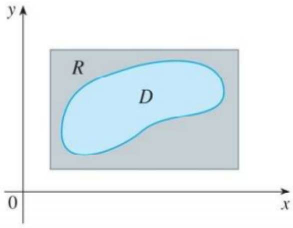
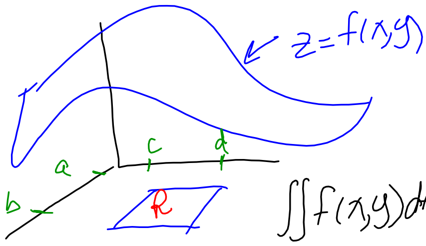
Section 15.2 Double Integrals over General Regions

For single integrals, the region over which we integrate is always an interval $[a, b]$. For double integrals, we want to be able to integrate not just over rectangles $[a, b] \times [c, d]$ but also over regions D of a more general shape. Suppose D is a bounded region, which means that D can be enclosed in a rectangular region.





$a \le x \le b$
$c \le y \le d$

$$\iint_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx$$

or

$$= \int_c^d \int_a^b f(x,y)\,dx\,dy$$

There are two types of plane regions:

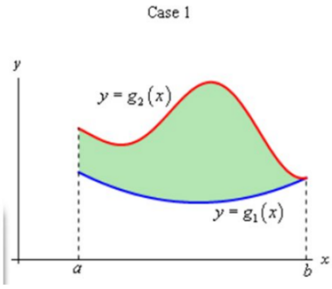
Type I: A plane region D is said to be of type I if it lies between the graphs of two continuous functions of x, that is $D = \{(x, y) | a \le x \le b, g_1(x) \le y \le g_2(x)\}$.

If f is continuous on a type I region $D = \{(x, y) | a \le x \le b, g_1(x) \le y \le g_2(x)\}$, then

$$\iint_R f(x,y)\,dA = \int_a^b \int_{g_1(x)=y}^{g_2(x)=y} f(x,y)\,dy\,dx$$

"x between numbers"

"y between functions of x"



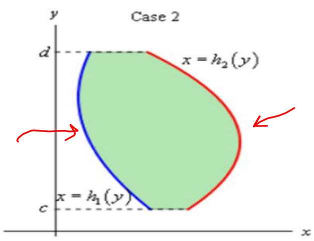
Type II: A plane region D is said to be of type II if it lies between the graphs of two continuous functions of y, that is $D = \{(x, y) | c \le y \le d, h_1(y) \le x \le h_2(y)\}$.

If f is continuous on a type II region $D = \{(x, y) | c \le y \le d, h_1(y) \le x \le h_2(y)\}$, then

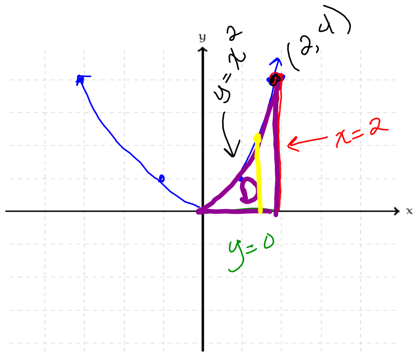
$$\iint_D f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$$

"y between numbers"

"x between functions"



Example 1: Sketch the region of integration and evaluate $\iint_D x\cos y\, dA$ where D is the region bounded by $y=0$, $y=x^2$ and $x=2$.



describe D: $0\le x\le 2$, $0\le y\le x^2$

Type I.

$$\iint_R x\cos y\, dA = \int_0^2\left[\int_{0=y}^{x^2=y} x\cos y\, dy\right]dx$$

$$=\int_0^2 x\sin y\Big|_{y=0}^{y=x^2} dx$$

$$=\int_0^2 \left(x\sin(x^2)-0\right)dx$$

$u=x^2$ $\quad x=2, u=4$; $\quad x=0, u=0$

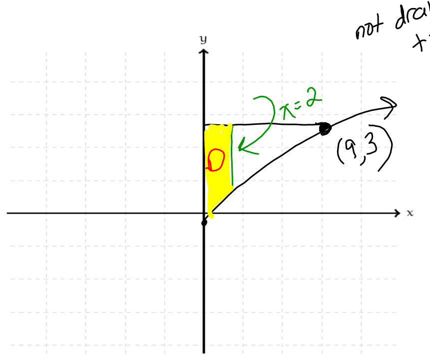
$du=2x\,dx$

$$\frac{1}{2}\int_0^4 \sin u\, du$$

$$-\frac{1}{2}(\cos u)\Big|_0^4 \to \boxed{-\frac{1}{2}[\cos(4)-1]}$$

Example 2: Sketch the region of integration and evaluate $\int_0^2\int_{\sqrt{x}}^3 (x^2+y)\, dy dx$.

not drawn to scale



$$\boxed{\begin{array}{c} 0\le x\le 2 \\ \sqrt{x}\le y\le 3\end{array}}$$

$y=\sqrt{x}$

$y=3$

$$\int_0^2\left[\int_{\sqrt{x}}^3 (x^2+y)\,dy\right]dx$$

$$\int_0^2\left[\left(x^2y+\frac{y^2}{2}\right)\right]\Big|_{y=\sqrt{x}}^{y=3} dx$$

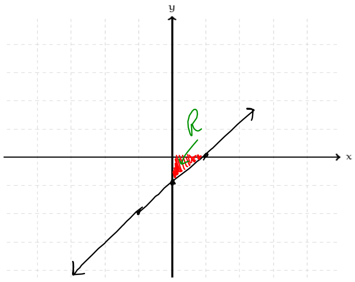
$x^2x^{\frac{1}{2}}=x^{\frac{5}{2}}$

$$\int_0^2\left[3x^2+\frac{9}{2}-\left(x^2\sqrt{x}+\frac{x}{2}\right)\right]dx$$

$$=\left(x^3+\frac{9}{2}x-\frac{2}{7}x^{\frac{7}{2}}-\frac{x^2}{4}\right)\Big|_0^2$$

$$=8+\frac{9}{2}(2)-\frac{2}{7}\left(2^{\frac{7}{2}}\right)-1$$

Example 3: Sketch the region of integration and evaluate $\int_0^1 \int_{x-1}^0 \frac{2y}{x+1}\, dy dx$.



$0 \le x \le 1$

$x-1 \le y \le 0 \leftarrow$ $y = x-1$, $y = 0$

$(2y)\left(\frac{1}{x+1}\right)$

$$\int_0^1 \left[\int_{x-1}^0 \frac{2y}{x+1}\, dy \right] dx$$

$$\int_0^1 \left[\frac{y^2}{x+1} \Bigg|_{y=x-1}^{y=0} \right] dx$$

$$\int_0^1 \left[- \frac{(x-1)^2}{x+1} \right] dx$$

$u-1$

$x=0,\ u=1$

$x=1,\ u=2$

$u = x+1 \longrightarrow x = u-1$

$du = dx$ $\quad x-1 = u-2$

$$-\int_1^2 \frac{(u-2)^2}{u}\, du$$

$$= -\int_1^2 \frac{u^2 - 4u + 4}{u}\, du$$

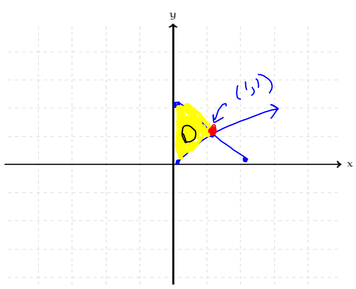
$$= -\int_1^2 \left(u - 4 + \frac{4}{u}\right) du$$

$$= -\left[\frac{u^2}{2} - 4u + 4\ln|u| \right]\Bigg|_1^2$$

$$= -\left[2 - 8 + 4\ln(2) - \left(\frac{1}{2} - 4\right)\right]$$

Example 4: Sketch the region of integration and evaluate $\iint_D (x^2 - 2xy)\, dA$, where

$D = \{(x,y) | 0 \le x \le 1, \sqrt{x} \le y \le 2 - x\}$.



$0 \le x \le 1$

$\boxed{\sqrt{x} \le y \le 2-x}$

$y = \sqrt{x}$

$y = 2-x$

$$\int_0^1 \left[\int_{\sqrt{x} = y}^{2-x = y} (x^2 - 2xy)\, dy \right] dx$$

$$\int_0^1 \left[x^2 y - \cancel{2} x \frac{y^2}{\cancel{2}} \right] \Bigg|_{y=\sqrt{x}}^{y=2-x} dx$$

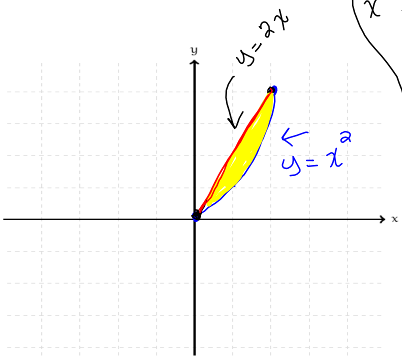
$$\int_0^1 \left[x^2(2-x) - x(2-x)^2 - \left(x^2\sqrt{x} - x^2\right) \right] dx$$

$$\boxed{\frac{1}{21}}$$

Example 5: Evaluate $\iint_D (x^2+y^2)\,dA$, where

$D = \{(x,y) | 0 \le x \le 2, x^2 \le y \le 2x\}$ as both a type I and type II region.

x between numbers
y between two functions of x



Type I:

$0 \le x \le 2$

$x^2 \le y \le 2x$

$y = x^2$ $\quad x^2 = 2x$

$y = 2x$ $\quad x^2 - 2x = 0$

$x(x-2) = 0$

$x=0, x=2$

$$\int_0^2 \left[\int_{x^2}^{2x} (x^2+y^2)\,dy\right] dx$$

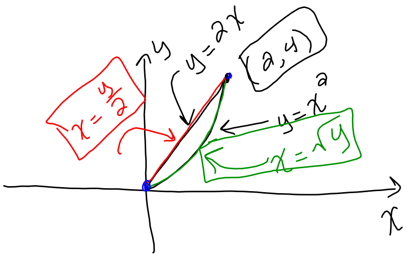
$$\int_0^2 \left[x^2 y + \frac{y^3}{3}\right]\Bigg|_{y=x^2}^{y=2x} dx$$

$$\int_0^2 \left[x^2(2x) + \frac{(2x)^3}{3} - \left(x^4 + \frac{x^6}{3}\right)\right] dx$$

$$\boxed{\frac{216}{35}}$$

Type II

"y between numbers
x between functions."



$0 \le y \le 4$

$\frac{y}{2} \le x \le \sqrt{y}$

$$\int_c^d \int_a^b h(x)g(y)\,dx\,dy$$

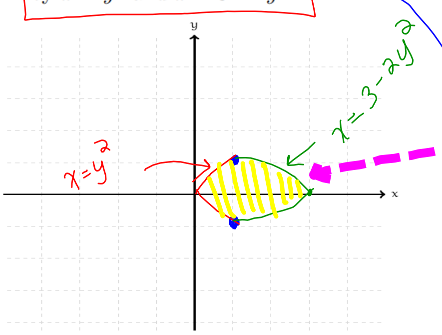
$$\int_c^d g(y)\,dy \int_a^b h(x)\,dx$$

$$\int_0^4 \int_{\frac{y}{2}}^{\sqrt{y}} (x^2+y^2)\,dx\,dy$$

Example 6: Set up but do not evaluate a type II integral for $\iint_D (y^2 - x)\,dA$, where D is the region bounded by $x = y^2$ and $x = 3 - 2y^2$.

$y^2 = 3 - 2y^2$

$3y^2 = 3$ $\quad \boxed{y = \pm 1}$



Type II $\quad -1 \le y \le 1$

$y^2 \le x \le 3 - 2y^2$

$$\int_{-1}^{1} \int_{y^2}^{3-2y^2} (y^2 - x)\,dx\,dy$$

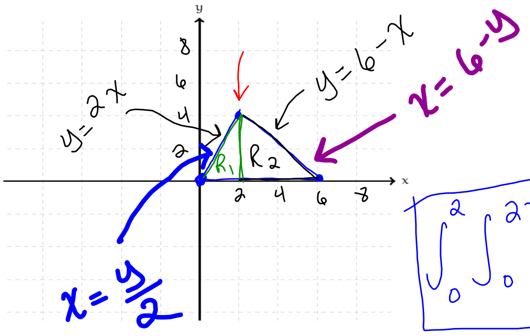
Example 7: Set up but do not evaluate both a type I and type II integral for $\iint_D ye^x\, dA$, where D is the triangular region with vertices $(0,0)$, $(2,4)$ and $(6,0)$.

Type 1: R_1: $0 \le x \le 2$

$0 \le y \le 2x$

R_2: $2 \le x \le 6$

$0 \le y \le 6-x$



$$\boxed{\int_0^2 \int_0^{2x} ye^x\, dy\, dx + \int_2^6 \int_0^{6-x} ye^x\, dy\, dx}$$

Type 2: $0 \le y \le 4$

$\frac{y}{2} \le x \le 6-y$

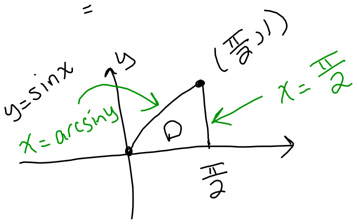
$$\int_0^4 \int_{\frac{y}{2}}^{6-y} ye^x\, dx\, dy$$

Example 8: Sketch the region of integration and change the order of integration.

(i) $\int_0^{\pi/2} \int_0^{\sin x} f(x,y)\, dy dx$

$0 \le x \le \frac{\pi}{2}$

$\boxed{0 \le y \le \sin x} \rightarrow$ $y=0$, $y=\sin x$



Type II

$0 \le y \le 1$

$\arcsin y \le x \le \frac{\pi}{2}$

$$\int_0^1 \int_{\arcsin y}^{\frac{\pi}{2}} f(x,y)\, dx\, dy$$

(ii) $\int_0^1 \int_{y^2}^{2-y} f(x,y)\, dx dy$

$0 \le y \le 1$

$\boxed{y^2 \le x \le 2-y}$

$x = y^2$

$x = 2-y$

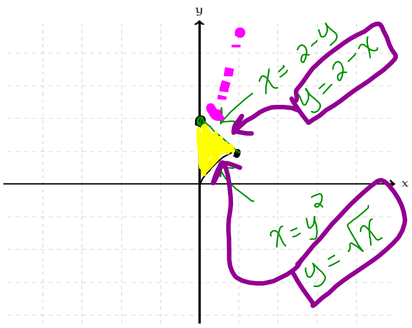
$y^2 = 2-y$

$y^2 + y - 2 = 0$

$(y+2)(y-1) = 0$

$\cancel{y=-2}$, $\boxed{y=1}$

not in domain



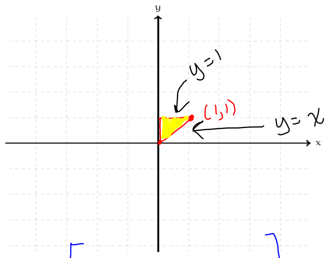
Type I:

$0 \le x \le 1$

$\sqrt{x} \le y \le 2-x$

$$\int_0^1 \int_{\sqrt{x}}^{2-x} f(x,y)\, dy\, dx$$

Example 9: Evaluate $\int_0^1 \int_x^1 \sin(y^2)\, dy dx$



change limits of integration!

$0 \le y \le 1$

$0 \le x \le y$

convert to $dx\,dy$

$$\int_0^1 \left[\int_0^y \sin(y^2)\, dx \right] dy \to \int_0^1 \left[x \sin(y^2) \Big|_{x=0}^{x=y} \right] dy$$

$u = y^2$; $y=1, u=1$; $y=0, u=0$

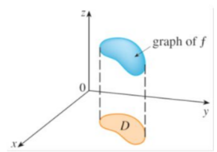
$du = 2y\,dy$

$$\int_0^1 \left[y \sin(y^2) \right] dy$$

$$\frac{1}{2}\int_0^1 \sin u\, du = \frac{-1}{2} \cos u \Big|_0^1$$

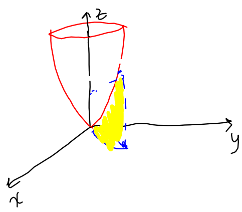
$$= \boxed{\frac{-1}{2}\left(\cos(1) - 1\right)}$$

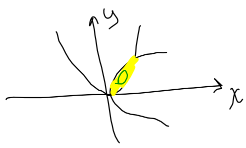
Theorem: If $f(x, y) \geq 0$ and f is continuous on the region R, then the volume V of the solid that lies above R and under the surface $f(x, y)$ is

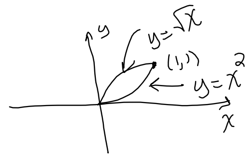


$$V = \iint_R f(x, y)\, dA$$

Example 10: Find the volume of the solid under the paraboloid $z = x^2 + y^2$ above the region bounded by $y = x^2$ and $x = y^2$.







Type I: $0 \le x \le 1$

$x^2 \le y \le \sqrt{x}$

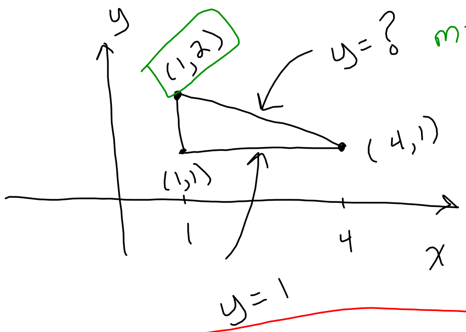
$$V = \int_0^1 \int_{x^2}^{\sqrt{x}} (x^2 + y^2)\, dy dx$$

$$= \int_0^1 \left[x^2 y + \frac{y^3}{3} \right] \Big|_{y=x^2}^{y=\sqrt{x}}$$

$$= \int_0^1 \left[x^2 \sqrt{x} + \frac{(\sqrt{x})^3}{3} - \left(x^4 + \frac{y^6}{3} \right) \right] dx$$

$$\boxed{\frac{6}{35}}$$

Example 11: Find the volume of the solid under the surface $z = xy$ and above the triangle with vertices $(1, 1)$, $(4, 1)$ and $(1, 2)$. [Set up only. Do not evaluate]



$m = -\frac{1}{3}$

$y - 2 = -\frac{1}{3}(x-1)$

$\boxed{y = -\frac{1}{3}x + \frac{7}{3}}$

$dy\,dx$

x between numbers

y between functions of x.

$1 \le x \le 4$

$1 \le y \le -\frac{1}{3}x + \frac{7}{3}$

$$\boxed{\int_1^4 \int_1^{-\frac{1}{3}x + \frac{7}{3}} xy\,dy\,dx}$$

NOTE: If we integrate the constant function $f(x, y) = 1$ over a region D, we get the area of D, that is

$$\iint_D 1\,dA = A(D)$$