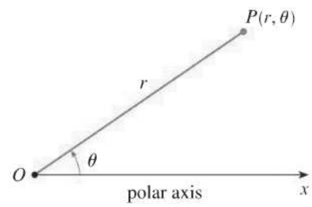
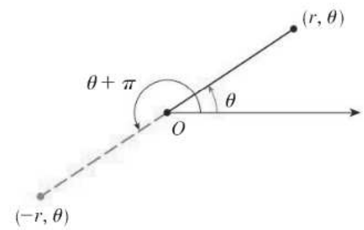
Section 15.3 Double Integrals in Polar Coordinates

In sections 15.1-15.2, we learned how to evaluate integrals of the form $\iint_R f(x,y)\,dA$. We had to determine the order of integration, and dA became either $dxdy$ or $dydx$. Such integrals are evaluated in *rectangular* coordinates. In this section, we learn how to integrate over regions that are *circular* in nature.

Recall: If $P(x,y)$ is a point in the xy-plane, we can represent the point P in polar form: Let r be the distance from O to P and let θ be the angle between the polar axis and the line OP. Then the point P is represented by the ordered pair (r,θ), and r, θ are called the **polar coordinates** of P.





Connecting polar coordinates with rectangular coordinates:

a.) $x = r\cos(\theta)$, $y = r\sin(\theta)$

b.) $\tan(\theta) = \dfrac{y}{x}$, thus $\theta = \arctan\left(\dfrac{y}{x}\right)$.

c.) $x^2 + y^2 = r^2$

Example 1: Find the cartesian coordinates of the polar point $\left(2, \dfrac{2\pi}{3}\right)$.

Example 2: Find the polar coordinates of the rectangular point $\left(\sqrt{3}, -1\right)$.

Example 3: Find a cartesian equation for the curve described by $r = 2\sin\theta$.

Example 4: Find a polar equation for $y = 1 + 3x$.

Double Integrals in Polar Coordinates: We use this method when we are integrating $z = f(x, y)$ over a region R in the xy-plane, where R is circular in nature.

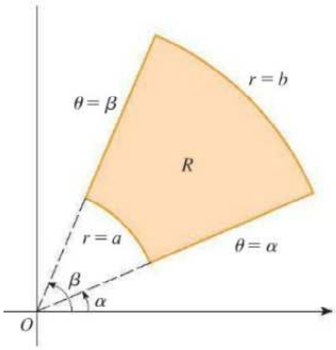
Suppose we want to evaluate a double integral $\iint_R f(x, y)\, dA$, where R is the region bounded by the unit circle $x^2 + y^2 = 1$.

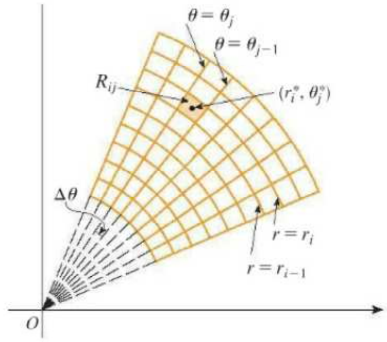
a.) Describe R as a Type I region

b.) Describe R as a Type II region

c.) Describe R as a Polar region

The region shown below is called a **polar rectangle**. $R = \{(r,\theta)|0 \le a \le r \le b, \alpha \le \theta \le \beta\}$.





Recall the area of a sector of a circle with radius r and central angle θ is $A = \frac{1}{2}r^2\theta$. If we look at the partition of the polar rectangle above, and consider the polar rectangle R, we can find the area of R by subtracting the areas of the two sectors, each of which has central angle $\Delta\theta = \beta - \alpha$. The area of the region R is

$\Delta A = \frac{1}{2}b^2\Delta\theta - \frac{1}{2}a^2\Delta\theta = \frac{1}{2}(b+a)(b-a)\Delta\theta = r^*\Delta r\Delta\theta$, where r^* is the average of $r = b$ and $r = a$.

Change to Polar Coordinates in a Double Integral: If f is continuous on a polar rectangle R given by $0 \le a \le r \le b$, $\alpha \le \theta \le \beta$, where $0 \le \beta - \alpha \le 2\pi$, then

$$\iint_R f(x,y)\,dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta) r\,dr\,d\theta$$

Example 5: Evaluate $\iint_R x\,dA$, where R is the region in the second quadrant bounded by the circle $x^2 + y^2 = 1$ and $y = 0$.

Example 6: Evaluate $\iint_R (3x + 4y^2)\, dA$, where R is the region bounded by the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

Example 7: Evaluate $\iint_R 7y\, dA$, where R is the region in the first quadrant enclosed by the by the circle $x^2 + y^2 = 9$ and the lines $y = 0$ and $y = x$.

Example 8: Evaluate $\int_{-3}^{3}\int_{0}^{\sqrt{9-x^2}} e^{x^2+y^2}\,dydx$ by converting to polar coordinates.

Example 9: Evaluate $\int_{0}^{4}\int_{0}^{\sqrt{4x-x^2}} \sqrt{x^2+y^2}\,dydx$ by converting to polar coordinates.

Recall: If $f(x, y) \geq 0$ and f is continuous on the region R, then the volume V of the solid that lies above R and under the surface $f(x, y)$ is $V = \iint_R f(x, y)\, dA$

Example 10: Find the volume of the solid bounded by the plane $z = 0$ and the paraboloid $z = 1 - x^2 - y^2$.

Example 11: Find the volume of the solid that lies above the xy-plane, below the sphere $x^2 + y^2 + z^2 = 81$ and inside the cylinder $x^2 + y^2 = 4$.

Example 12: Find the volume under the cone $z = \sqrt{x^2 + y^2}$ and above the ring $4 \leq x^2 + y^2 \leq 25$.

Example 13: Find the volume of the solid bounded by the paraboloids $z = 20 - x^2 - y^2$ and $z = 4x^2 + 4y^2$.