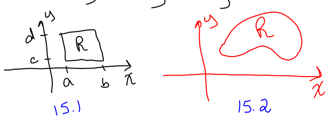
In section 15.1, 15.2, we were integrating regions over a region R in the xy plane.

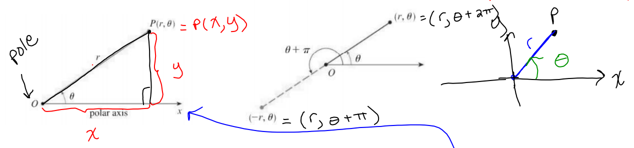


Section 15.3 Double Integrals in Polar Coordinates

In sections 15.1-15.2, we learned how to evaluate integrals of the form $\iint_R f(x,y)\,dA$. We had to determine the order of integration, and dA became either $dxdy$ or $dydx$. Such integrals are evaluated in *rectangular* coordinates. In this section, we learn how to integrate over regions that are *circular* in nature.

Recall: If $P(x,y)$ is a point in the xy-plane, we can represent the point P in polar form: Let r be the distance from O to P and let θ be the angle between the polar axis and the line OP. Then the point P is represented by the ordered pair (r,θ), and r, θ are called the **polar coordinates** of P.

Polar point: (r, θ)



Connecting polar coordinates with rectangular coordinates:

a.) $x = r\cos(\theta)$, $y = r\sin(\theta)$

b.) $\tan(\theta) = \frac{y}{x}$, thus $\theta = \arctan\left(\frac{y}{x}\right)$.

c.) $x^2 + y^2 = r^2$

$\cos\theta = \frac{x}{r}$ $\quad x = r\cos\theta$

$\sin\theta = \frac{y}{r}$ $\quad y = r\sin\theta$

Example 1: Find the cartesian coordinates of the polar point $\left(2, \frac{2\pi}{3}\right)$.

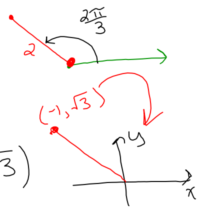
$r = 2$

$\theta = \frac{2\pi}{3}$

$x = r\cos\theta = 2\cos\frac{2\pi}{3} = -1$

$y = r\sin\theta = 2\sin\frac{2\pi}{3} = \sqrt{3}$

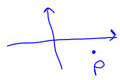
Cartesian point $(-1, \sqrt{3})$



Example 2: Find the polar coordinates of the rectangular point $(\sqrt{3}, -1)$.

$x = \sqrt{3}$, $y = -1$

Q IV



$x^2 + y^2 = r^2$

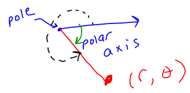
$3 + 1 = r^2$

$r^2 = 4$

$r = \pm 2$

Some choices for polar point.

$r = 2$



Find θ: $\tan\theta = \frac{y}{x} = \frac{-1}{\sqrt{3}}$

$\theta = \arctan\left(-\frac{1}{\sqrt{3}}\right)$

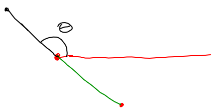
recall: $\tan\frac{\pi}{6} = \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}}$

$\tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$

if $r = 2$, $\theta = -\frac{\pi}{6}$, $\theta = 2\pi - \frac{\pi}{6} = \frac{11\pi}{6}$

$\boxed{\left(2, -\frac{\pi}{6}\right), \left(2, \frac{11\pi}{6}\right)}$

$r = -2$



$\theta = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$

$\left(-2, \frac{5\pi}{6}\right) = \left(2, -\frac{\pi}{6}\right)$

Example 3: Find a cartesian equation for the curve described by $r = 2\sin\theta$.

- A polar equation, involves r's & θ's
- A cartesian equation involves x's & y's.

$x = r\cos\theta$

$y = r\sin\theta$

$x^2 + y^2 = r^2$

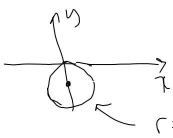
$r = 2\sin\theta$

$\underbrace{r^2}_{x^2+y^2} = 2\underbrace{r\sin\theta}_{y}$

$x^2 + y^2 - 2y = 0$ $\quad \boxed{x^2 + y^2 = 2y}$

$x^2 + y^2 - 2y + 1 = 1$

$x^2 + (y-1)^2 = 1$

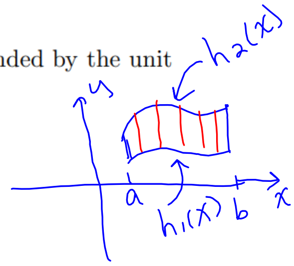


Example 4: Find a polar equation for $y = 1 + 3x$.

$$\boxed{r\sin\theta = 1 + 3r\cos\theta}$$

Double Integrals in Polar Coordinates: We use this method when we are integrating $z = f(x, y)$ over a region R in the xy-plane, where R is circular in nature.

Suppose we want to evaluate a double integral $\iint_R f(x, y)\, dA$, where R is the region bounded by the unit circle $x^2 + y^2 = 1$. $\rightarrow y = \pm\sqrt{1-x^2}$

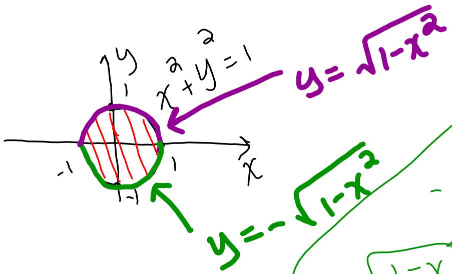


a.) Describe R as a Type I region

R is type I if

$a \le x \le b$

$h_1(x) \le y \le h_2(x)$



R

$-1 \le x \le 1$

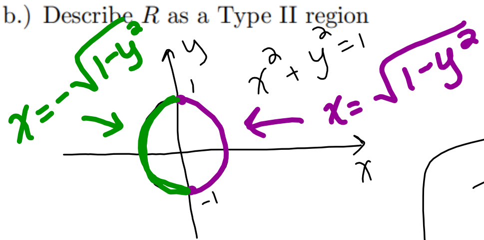
$-\sqrt{1-x^2} \le y \le \sqrt{1-x^2}$

b.) Describe R as a Type II region

R is type II if

$c \le y \le d$

$g_1(y) \le x \le g_2(y)$

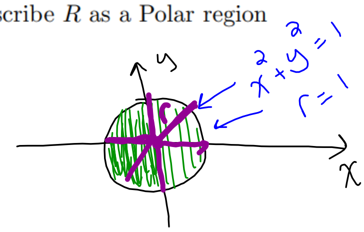


Type 2

$-1 \le y \le 1$

$-\sqrt{1-y^2} \le x \le \sqrt{1-y^2}$

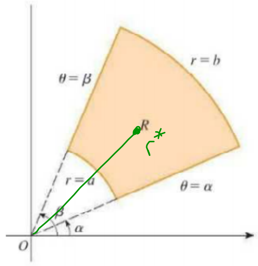
c.) Describe R as a Polar region

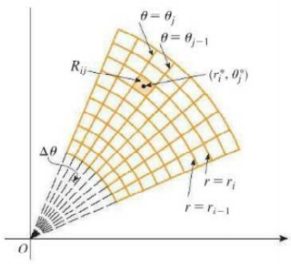


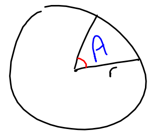
$0 \le r \le 1$

$0 \le \theta \le 2\pi$

The region shown below is called a **polar rectangle**. $R = \{(r,\theta) | 0 \le a \le r \le b, \alpha \le \theta \le \beta\}$.







Recall the area of a sector of a circle with radius r and central angle θ is $\boxed{A = \frac{1}{2}r^2\theta.}$ If we look at the partition of the polar rectangle above, and consider the polar rectangle R, we can find the area of R by subtracting the areas of the two sectors, each of which has central angle $\Delta\theta = \beta - \alpha$. The area of the region R is

$\Delta A = \frac{1}{2}b^2\Delta\theta - \frac{1}{2}a^2\Delta\theta = \frac{1}{2}(b+a)\underbrace{(b-a)}_{\Delta r}\Delta\theta = r^*\Delta r\Delta\theta$, where r^* is the average of $r=b$ and $r=a$.

$$\iint_{\boxed{R}} f(x,y)\, dA \quad dA \to r\,dr\,d\theta$$

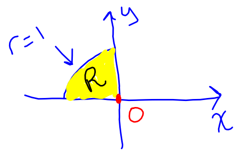
$\boxed{R} \to$ bounds on θ
bounds on r

Change to Polar Coordinates in a Double Integral: If f is continuous on a polar rectangle R given by $0 \le a \le r \le b$, $\alpha \le \theta \le \beta$, where $0 \le \beta - \alpha \le 2\pi$, then

$$\boxed{\iint_R f(x,y)\, dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\underbrace{r\,dr\,d\theta}_{dA}}$$

R = circular in the xy plane

Example 5: Evaluate $\iint_R x\, dA$, where R is the region in the second quadrant bounded by the circle $x^2+y^2=1$ and $y=0$.



in Polar: $0 \le r \le 1$ $\quad x = r\cos\theta$

$\frac{\pi}{2} \le \theta \le \pi$

$$\iint_R x\, dA = \int_{\frac{\pi}{2}}^{\pi}\int_0^1 r\cos\theta\, r\, dr\, d\theta$$

$$= \int_{\frac{\pi}{2}}^{\pi}\int_0^1 r^2\cos\theta\, dr\, d\theta$$

Fubini!!

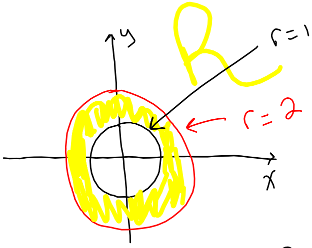
$$= \int_{\frac{\pi}{2}}^{\pi}\cos\theta\, d\theta \int_0^1 r^2\, dr$$

$$= \left[\sin\theta \Big|_{\frac{\pi}{2}}^{\pi}\right]\left[\frac{r^3}{3}\Big|_0^1\right]$$

$$= (0-1)\left(\frac{1}{3}\right) = \boxed{-\frac{1}{3}}$$

Example 6: Evaluate $\iint_R (3x+4y^2)\,dA$, where R is the region bounded by the circles $x^2+y^2=1$ and $x^2+y^2=4$.

$r\,dr\,d\theta$



$1 \le r \le 2, \quad 0 \le \theta \le 2\pi$

$$\int_0^{2\pi}\int_1^2 \left(3r\cos\theta + 4r^2\sin^2\theta\right) r\,dr\,d\theta$$

$$\int_0^{2\pi}\int_1^2 \left(3r^2\cos\theta + 4r^3\sin^2\theta\right) dr\,d\theta$$

$$\int_0^{2\pi}\left[r^3\cos\theta + r^4\sin^2\theta \Big|_{r=1}^{r=2} \right] d\theta$$

$$\int_0^{2\pi}\left[8\cos\theta + 16\sin^2\theta - \left(\cos\theta + \sin^2\theta\right)\right] d\theta$$

$$\int_0^{2\pi}\left(7\cos\theta + \boxed{15\sin^2\theta}\right) d\theta$$

$\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$

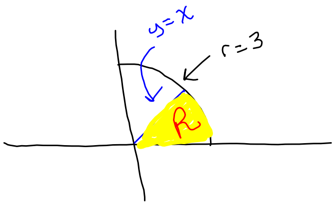
$\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$

$\frac{15}{2}(1-\cos 2\theta)$

$$\left[7\sin\theta + \frac{15}{2}\left(\theta - \frac{1}{2}\sin 2\theta\right)\right]\Big|_0^{2\pi} = \left[0 + \frac{15}{2}(2\pi)\right]$$

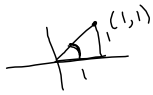
$$= \boxed{15\pi}$$

Example 7: Evaluate $\iint_R 7y\,dA$, where R is the region in the first quadrant enclosed by the by the circle $x^2+y^2=9$ and the lines $y=0$ and $y=x$.



$0 \le r \le 3$

$0 \le \theta \le \frac{\pi}{4}$



$$\iint_R 7y\,dA = \int_0^{\frac{\pi}{4}}\int_0^3 (7r\sin\theta)\,r\,dr\,d\theta$$

$$7\left[\int_0^{\frac{\pi}{4}} \sin\theta\,d\theta\right]\left[\int_0^3 r^2\,dr\right]$$

$$7\left[-\cos\theta\Big|_0^{\frac{\pi}{4}}\right]\left[\frac{r^3}{3}\Big|_0^3\right]$$

$$-7\left(\cos\frac{\pi}{4} - \cos 0\right)(9)$$

$$= \boxed{-7\left(\frac{\sqrt{2}}{2} - 1\right)(9)}$$

Example 8: Evaluate $\int_{-3}^{3}\int_{0}^{\sqrt{9-x^2}} e^{x^2+y^2}\,dy\,dx$ by converting to polar coordinates.

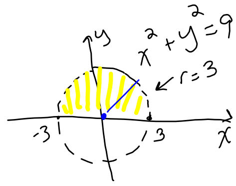
$$\boxed{-3 \le x \le 3}$$
$$\boxed{0 \le y \le \sqrt{9-x^2}}$$

$y = 0$

$y = \sqrt{9-x^2}$

$y^2 = 9 - x^2$

$x^2 + y^2 = 9$



$0 \le r \le 3$

$0 \le \theta \le \pi$

$$\int_0^{\pi}\int_0^3 e^{r^2}\, r\,dr\,d\theta$$

$u = r^2$

$du = 2r\,dr$

$\frac{1}{2}\int e^u\,du$

$\frac{1}{2}e^u$

$\frac{1}{2}e^{r^2}$

$$\left(\int_0^{\pi} d\theta\right)\left(\int_0^3 r e^{r^2}\,dr\right)$$

$$\left(\theta\Big|_0^{\pi}\right)\left(\frac{1}{2}e^{r^2}\Big|_0^3\right)$$

$$\boxed{(\pi)\left(\frac{1}{2}e^9 - \frac{1}{2}\right)}$$

Example 9: Evaluate $\int_{0}^{4}\int_{0}^{\sqrt{4x-x^2}} \sqrt{x^2+y^2}\,dy\,dx$ by converting to polar coordinates.

$0 \le x \le 4$

$$\boxed{0 \le y \le \sqrt{4x-x^2}}$$

$y = 0$

$y = \sqrt{4x-x^2}$

$y^2 = 4x - x^2$

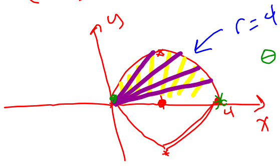
$x^2 + y^2 = 4x$

$r^2 = 4r\cos\theta$

$r = 4\cos\theta$

$x^2 - 4x + 4 + y^2 = 0 + 4$

$(x-2)^2 + y^2 = 4$



$0 \le r \le 4\cos\theta$

$0 \le \theta \le \frac{\pi}{2}$

$$\int_0^{\frac{\pi}{2}}\int_0^{4\cos\theta} r\; r\,dr\,d\theta \qquad (r^2)$$

$$\int_0^{\frac{\pi}{2}}\left[\frac{r^3}{3}\Big|_{r=0}^{r=4\cos\theta}\right]d\theta = \frac{1}{3}\int_0^{\frac{\pi}{2}} 64\cos^3\theta\,d\theta$$

$$= \frac{64}{3}\int_0^{\frac{\pi}{2}}\cos^2\theta\cos\theta\,d\theta$$

$u = \sin\theta$; $\theta = \frac{\pi}{2}, u = 1$; $\theta = 0, u = 0$

$du = \cos\theta\,d\theta$

$$= \frac{64}{3}\int_0^{\frac{\pi}{2}}(1-\sin^2\theta)\cos\theta\,d\theta$$

$$= \frac{64}{3}\int_0^1 (1-u^2)\,du$$

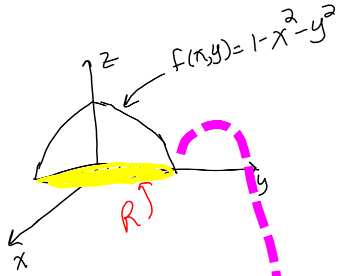
$$= \frac{64}{3}\left(u - \frac{u^3}{3}\right)\Big|_0^1$$

$$= \frac{64}{3}\left(1 - \frac{1}{3}\right)$$

$$= \boxed{\frac{64}{3}\left(\frac{2}{3}\right)}$$

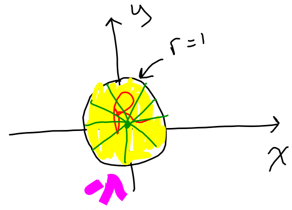
Recall: If $f(x,y) \geq 0$ and f is continuous on the region R, then the volume V of the solid that lies above R and under the surface $f(x,y)$ is $V = \iint_R f(x,y)\, dA$

Example 10: Find the volume of the solid bounded by the plane $z=0$ and the paraboloid $z = 1 - x^2 - y^2$.



$0 = 1 - x^2 - y^2$

$\boxed{x^2+y^2=1}$



$0 \leq r \leq 1$

$0 \leq \theta \leq 2\pi$

in polar,

$f(x,y) = 1 - x^2 - y^2$

$= 1 - r^2$

$$V = \int_0^{2\pi}\int_0^1 (1-r^2)\, r\, dr\, d\theta$$

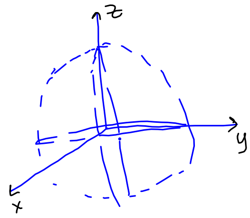
$$= \left(\int_0^{2\pi} d\theta\right)\left(\int_0^1 (r - r^3)\, dr\right)$$

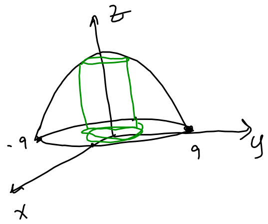
$$= \left[\theta\Big|_0^{2\pi}\right]\left[\left(\frac{r^2}{2} - \frac{r^4}{4}\right)\Big|_0^1\right] \rightarrow \boxed{(2\pi)\left(\frac{1}{2}-\frac{1}{4}\right)}$$

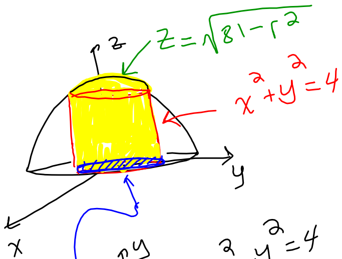
$z^2 = 81 - x^2 - y^2$

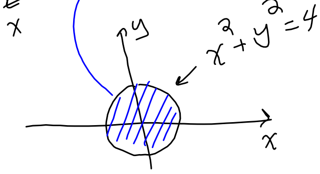
$= 81 - r^2$

Example 11: Find the volume of the solid that lies above the xy-plane, below the sphere $x^2+y^2+z^2=81$ and inside the cylinder $x^2+y^2=4$.









$$V = \int_0^{2\pi}\int_0^2 \sqrt{81-r^2}\; r\, dr\, d\theta$$

$0 \leq r \leq 2$

$0 \leq \theta \leq 2\pi$

Fubini $\displaystyle \int_0^{2\pi} d\theta \int_0^2 r\sqrt{81-r^2}\, dr$

$u = 81 - r^2$

$du = -2r\, dr$

$-\frac{1}{2}\int u^{\frac{1}{2}}\, du = -\frac{1}{2}\cdot\frac{2}{3} u^{3/2}$

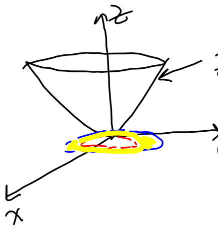
$= -\frac{1}{3}(81-r^2)^{3/2}$

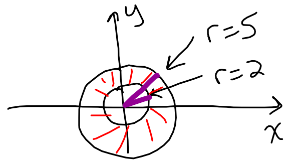
$$\left[\theta\Big|_0^{2\pi}\right]\left[-\frac{1}{3}(81-r^2)^{\frac{3}{2}}\Big|_0^2\right]$$

$$(2\pi)\left(-\frac{1}{3}\left(77^{3/2} - 81^{3/2}\right)\right) = -\frac{2\pi}{3}\left(77\sqrt{77} - 81\sqrt{81}\right)$$

$$= -\frac{2\pi}{3}\left(77\sqrt{77} - 81(9)\right)$$

Example 12: Find the volume under the cone $z = \sqrt{x^2+y^2}$ and above the ring $4 \le x^2 + y^2 \le 25$.





$2 \le r \le 5$

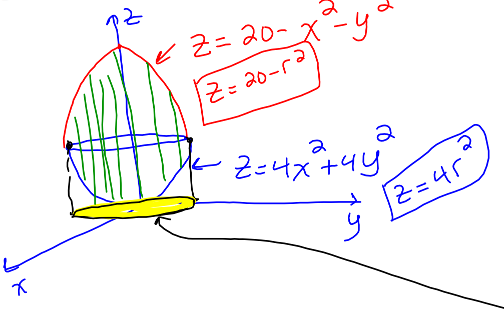
$0 \le \theta \le 2\pi$

$$\int_0^{2\pi}\int_2^5 r \, r\,dr\,d\theta = \int_0^{2\pi} d\theta \int_2^5 r^2\,dr$$

$$= \theta\Big|_0^{2\pi} \; \frac{r^3}{3}\Big|_2^5$$

$$= (2\pi)\left(\frac{125}{3} - \frac{8}{3}\right)$$

Example 13: Find the volume of the solid bounded by the paraboloids $z = 20 - x^2 - y^2$ and $z = 4x^2 + 4y^2$.

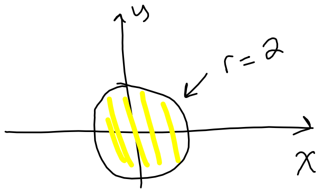


paraboloids intersect when

$$4x^2 + 4y^2 = 20 - x^2 - y^2$$

$$5x^2 + 5y^2 = 20$$

$$\boxed{x^2 + y^2 = 4}$$



$0 \le r \le 2$

$0 \le \theta \le 2\pi$

integrand is "Top - Bottom"

$20 - r^2 - (4r^2)$

$20 - 5r^2$

$$\int_0^{2\pi}\int_0^2 (20 - 5r^2)\, r\,dr\,d\theta$$

$$\int_0^{2\pi} d\theta \int_0^2 (20r - 5r^3)\,dr = 40\pi$$