Section 15.6 Triple Integrals

Just as we defined single integrals for functions of one variable and double integrals for functions of two variables, we now define triple integrals for functions of three variables.

Definition: The **Triple Integral** of f over the box $E = \{(x, y, z)|a \leq x \leq b, c \leq y \leq d, r \leq z \leq s\}$ is

$$\iiint_E f(x,y,z)\,dV = \iiint_E f(x,y,z)\,dxdydz$$

Fubini's Theorem for Triple Integrals: If f is continuous over the rectangular box $E = [a, b] \times [c, d] \times [r, s]$, then $\iiint_E f(x,y,z)\,dV$ can be computed in six different orders:

(i) $\int_r^s \int_c^d \int_a^b f(x,y,z)\,dxdydz$ (ii) $\int_c^b \int_r^s \int_a^b f(x,y,z)\,dxdzdy$ (iii) $\int_r^s \int_a^b \int_c^d f(x,y,z)\,dydxdz$

(iv) $\int_a^b \int_r^s \int_c^d f(x,y,z)\,dydzdx$ (v) $\int_c^d \int_a^b \int_r^s f(x,y,z)\,dzdxdy$ (vi) $\int_a^b \int_c^d \int_r^s f(x,y,z)\,dzdydx$

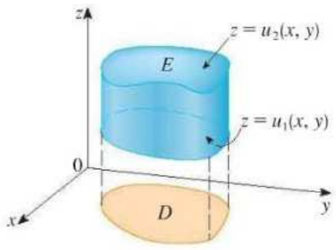
Also note the special case of Fubini applies and is similar to the way it worked with double integrals, provided all limits of integration are constants.

Example 1: Evaluate $\iiint_E xyz^2\,dV$ where $E = [0, 1] \times [-1, 2] \times [0, 3]$.

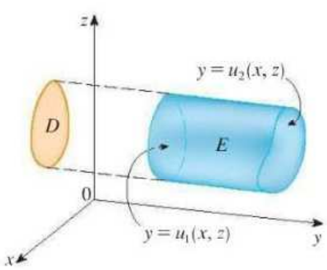
Example 2: $\int_0^1 \int_x^{x^2} \int_0^y xyz\,dzdydx$

Triple Integrals over a general bounded region E in three dimensional space:

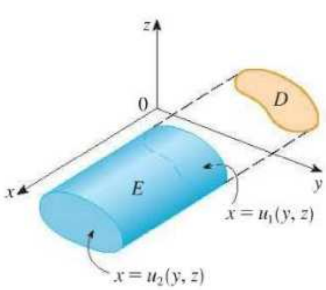
Type I: A solid region E is said to be of type I if it lies between the graphs of two continuous functions of x and y, that is $E = \{(x,y,z) | (x,y) \in D, u_1(x,y) \leq z \leq u_2(x,y)\}$ where D is the projection of E on the xy-plane. Notice that the upper bound of E is the surface $z = u_2(x,y)$ and the lower bound of E is the surface $z = u_1(x,y)$. Moreover, it can be shown that $\iiint_E f(x,y,z)\,dV = \iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\right] dA$



Type II: A solid region E is said to be of type II if it lies between the graphs of two continuous functions of x and z, that is $E = \{(x,y,z) | (x,z) \in D, u_1(x,z) \leq y \leq u_2(x,z)\}$ where D is the projection of E on the xz-plane. Notice that the right bound of E is the surface $y = u_2(x,z)$ and the left bound of E is the surface $y = u_1(x,z)$. Moreover, it can be shown that $\iiint_E f(x,y,z)\,dV = \iint_D \left[\int_{u_1(x,z)}^{u_2(x,z)} f(x,y,z)\,dy\right] dA$



Type III: A solid region E is said to be of type III if it lies between the graphs of two continuous functions of y and z, that is $E = \{(x,y,z) | (y,z) \in D, u_1(y,z) \leq x \leq u_2(y,z)\}$ where D is the projection of E on the yz-plane. Notice that the back surface of E is $x = u_1(y,z)$ and the front surface of E is the $x = u_2(y,z)$. Moreover, it can be shown that $\iiint_E f(x,y,z)\,dV = \iint_D \left[\int_{u_1(y,z)}^{u_2(y,z)} f(x,y,z)\,dx\right] dA$



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Example 3: Evaluate $\iiint_E x\,dV$ where E is bounded by the planes $x = 0$, $y = 0$, $z = 0$, $4x + 2y + z = 6$.

Example 4: Evaluate $\iiint_E xz\,dV$ where E is the solid tetrahedron with vertices $(0, 0, 0)$, $(0, 1, 0)$, $(1, 1, 0)$ and $(0, 1, 1)$.

Example 5: Evaluate $\iiint_E x\,dV$ where E is bounded by the paraboloid $x = 2y^2 + 2z^2$ and the plane $x = 2$.

Example 6: Evaluate $\iiint_E \sqrt{x^2 + z^2}\,dV$ where E is the region bounded by the paraboloid $y = x^2 + z^2$ and the plane $y = 4$.

Example 7: Set up but do not evaluate $\iiint_E x^2y^4z^4\,dV$ where E is the solid bounded by $4x^2 + z^2 = 4$, $y = 0$ and $y = z + 2$.

Note: We can use a triple integral to find the volume of a solid E using the fact that if $f(x, y, z) = 1$ for all points in E, then the volume of E is $\iiint_E dV$.

Example 8: Consider the tetrahedron enclosed by the three coordinate planes and the plane $2x + y + z = 4$. Set up but do not evaluate:

a.) a double integral that gives the volume of this solid.

b.) a triple integral that gives the volume of this solid.

Example 9: Find the volume of the solid bounded by the cylinder $x = y^2$ and the planes $z = 0$ and $x + z = 1$.

Example 10: Rewrite $\int_0^1 \int_0^{x^2} \int_0^y f(x, y, z)\, dzdydx$ as an iterated integral in a different order, integrating with respect to x, then z, then y by writing and graphing the projection of the solid E on the zy plane.