$\iint_R f(x,y)\,dA$, $z=f(x,y)$; R → xy plane

dA → $dy\,dx \to a \le x \le b,\ g_1(x) \le y \le g_2(x)$

$dx\,dy \to c \le y \le d,\ h_1(y) \le x \le h_2(y)$

$r\,dr\,d\theta \to$ R is a polar region in xy plane.

Section 15.6 Triple Integrals

Just as we defined single integrals for functions of one variable and double integrals for functions of two variables, we now define triple integrals for functions of three variables.

Definition: The **Triple Integral** of f over the box $E = \{(x,y,z) | a \le x \le b, c \le y \le d, r \le z \le s\}$ is

$$\iiint_E f(x,y,z)\,dV = \iiint_E f(x,y,z)\,dx\,dy\,dz$$

$w = f(x,y,z)$; E: region with volume

Fubini's Theorem for Triple Integrals: If f is continuous over the rectangular box $E = [a,b] \times [c,d] \times [r,s]$, then $\iiint_E f(x,y,z)\,dV$ can be computed in six different orders:

(i) $\int_r^s \int_c^d \int_a^b f(x,y,z)\,dx\,dy\,dz$ (ii) $\int_c^d \int_r^s \int_a^b f(x,y,z)\,dx\,dz\,dy$ (iii) $\int_r^s \int_a^b \int_c^d f(x,y,z)\,dy\,dx\,dz$

(iv) $\int_a^b \int_r^s \int_c^d f(x,y,z)\,dy\,dz\,dx$ (v) $\int_c^d \int_a^b \int_r^s f(x,y,z)\,dz\,dx\,dy$ (vi) $\int_a^b \int_c^d \int_r^s f(x,y,z)\,dz\,dy\,dx$

Also note the special case of Fubini applies and is similar to the way it worked with double integrals, provided all limits of integration are constants.

Example 1: Evaluate $\iiint_E xyz^2\,dV$ where $E = [0,1] \times [-1,2] \times [0,3]$. → $0 \le x \le 1$, $-1 \le y \le 2$, $0 \le z \le 3$

$dV \to dx\,dy\,dz$

$$\int_0^3 \int_{-1}^2 \int_0^1 xyz^2\,dx\,dy\,dz$$

Fubini!

$$\left(\int_0^3 z^2\,dz\right)\left(\int_{-1}^2 y\,dy\right)\left(\int_0^1 x\,dx\right)$$

$$\left(\frac{z^3}{3}\Big|_0^3\right)\left(\frac{y^2}{2}\Big|_{-1}^2\right)\left(\frac{x^2}{2}\Big|_0^1\right)$$

$$(9)\left(2-\frac{1}{2}\right)\left(\frac{1}{2}\right) = (9)\left(\frac{3}{2}\right)\left(\frac{1}{2}\right) = \boxed{\frac{27}{4}}$$

Example 2: $\int_0^1 \int_x^{x^2} \left(\int_0^y xyz\,dz\right) dy\,dx = \int_0^1 \int_x^{x^2} \left[xy\frac{z^2}{2} \Big|_{z=0}^{z=y} \right] dy\,dx$

$$= \frac{1}{2}\int_0^1 \int_x^{x^2} xy(y^2)\,dy\,dx$$

$$= \frac{1}{2}\int_0^1 \left(\int_x^{x^2} xy^3\,dy\right)dx$$

$$= \frac{1}{2}\int_0^1 x\frac{y^4}{4}\Big|_{y=x}^{y=x^2} dx$$

$$= \frac{1}{8}\int_0^1 \left[x(x^2)^4 - x(x^4) \right] dx$$

$$= \frac{1}{8}\int_0^1 \left(x^9 - x^5\right) dx$$

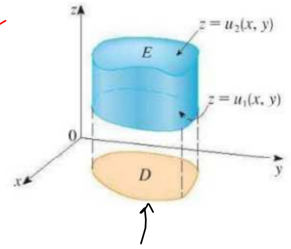
$$= \frac{1}{8}\left(\frac{x^{10}}{10} - \frac{x^6}{6}\right)\Big|_0^1$$

$$= \frac{1}{8}\left(\frac{1}{10} - \frac{1}{6}\right) = \boxed{-\frac{1}{120}}$$

Triple Integrals over a general bounded region E in three dimensional space:

Type I: A solid region E is said to be of type I if it lies between the graphs of two continuous functions of x and y, that is $E = \{(x,y,z)|(x,y) \in D, u_1(x,y) \le z \le u_2(x,y)\}$ where D is the projection of E on the xy-plane. Notice that the upper bound of E is the surface $z = u_2(x,y)$ and the lower bound of E is the surface $z = u_1(x,y)$. Moreover, it can be shown that $\iiint_E f(x,y,z)\,dV = \iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\right] dA$

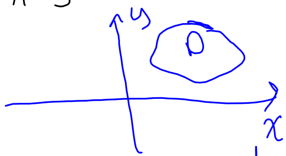
*



Type I triple integral

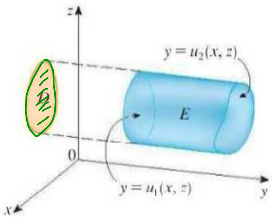
$dV = dz\, dA$, D is in the xy-plane

$u_1(x,y) \le z \le u_2(x,y)$



$dA = dx\,dy$ or $dy\,dx$

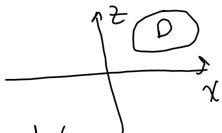
Type II: A solid region E is said to be of type II if it lies between the graphs of two continuous functions of x and z, that is $E = \{(x,y,z)|(x,z) \in D, u_1(x,z) \le y \le u_2(x,z)\}$ where D is the projection of E on the xz-plane. Notice that the right bound of E is the surface $y = u_2(x,z)$ and the left bound of E is the surface $y = u_1(x,z)$. Moreover, it can be shown that $\iiint_E f(x,y,z)\,dV = \iint_D \left[\int_{u_1(x,z)}^{u_2(x,z)} f(x,y,z)\,dy\right] dA$



Type II triple integral

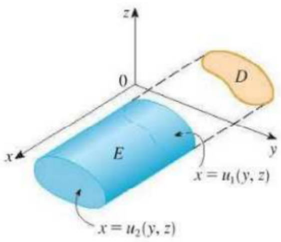
$dV = dy\, dA$, $u_1(x,z)\, y \le u_2(x,z)$

where D in in xz plane



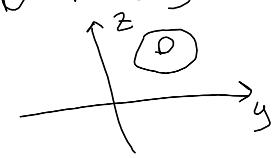
$dA = dx\,dz$ or $dz\,dx$

Type III: A solid region E is said to be of type III if it lies between the graphs of two continuous functions of y and z, that is $E = \{(x,y,z)|(y,z) \in D, u_1(y,z) \le x \le u_2(y,z)\}$ where D is the projection of E on the yz-plane. Notice that the back surface of E is $x = u_1(y,z)$ and the front surface of E is the $x = u_2(y,z)$. Moreover, it can be shown that $\iiint_E f(x,y,z)\,dV = \iint_D \left[\int_{u_1(y,z)}^{u_2(y,z)} f(x,y,z)\,dx\right] dA$



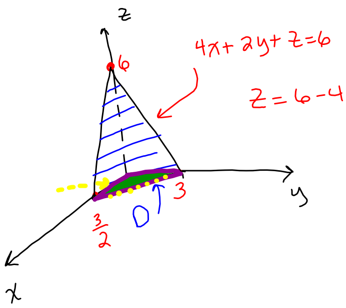
Type III triple integral is when

$dV = dx\, dA$, D in zy plane



$dA = dy\,dz$ or $dz\,dy$

Example 3: Evaluate $\iiint_E x\,dV$ where E is bounded by the planes $x = 0$, $y = 0$, $z = 0$, $4x + 2y + z = 6$.



describe E

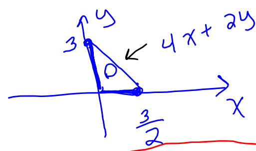
$dV = dz\,dA$ ← my choice!

$dV = dy\,dA$

$dV = dx\,dA$

$dz\,dA$: $0 \le z \le 6-4x-2y$

describe D: in xy plane



$y = 3-2x$

$x = \frac{6-2y}{4}$

$x = \frac{3}{2} - \frac{1}{2}y$

dA: $dx\,dy$: $0 \le y \le 3$, $0 \le x \le \frac{3}{2} - \frac{1}{2}y$

or

$dy\,dx$: $0 \le x \le \frac{3}{2}$, $0 \le y \le 3-2x$

$$\iiint_E x\,dV = \iiint x\,dz\,dA$$

$$= \int_0^{3/2}\int_0^{3-2x}\int_0^{6-4x-2y} x\,dz\,dy\,dx$$

$$= \int_0^{3/2}\int_0^{3-2x}\left[xz \Big|_{z=0}^{z=6-4x-2y} \right] dy\,dx$$

$$= \int_0^{3/2}\int_0^{3-2x}\left[x(6-4x-2y) \right] dy\,dx$$

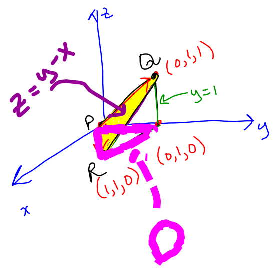
$$= \int_0^{3/2}\int_0^{3-2x} \left(6x - 4x^2 - 2xy\right) dy\,dx$$

$$= \int_0^{3/2} \left(6xy - 4x^2 y - xy^2 \Big|_{y=0}^{y=3-2x} \right) dx$$

$$= \int_0^{3/2} \left(6x(3-2x) - 4x^2(3-2x) - x(3-2x)^2 \right) dx$$

$$= \frac{27}{16}$$

Example 4: Evaluate $\iiint_E xz\, dV$ where E is the solid tetrahedron with vertices $(0,0,0)$, $(0,1,0)$, $(1,1,0)$ and $(0,1,1)$.



recall: The equation of a plane is

$n\cdot(r-r_0)=0$ $\quad \vec{n}$ = normal vector, r_0 = any point on the plane

This plane passes thru $P(0,0,0)$, $Q(0,1,1)$, $R(1,1,0)$

$\vec{n} = \vec{PQ}\times\vec{PR} = \langle 0,1,1\rangle \times \langle 1,1,0\rangle$

$$\begin{vmatrix} i & j & k \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{vmatrix} = \langle -1, 1, -1\rangle$$

$n = \langle -1,1,-1\rangle$, $r_0 = (0,0,0)$

$\vec{n}\cdot(\vec{r-r_0}) = 0$

$\langle -1,1,-1\rangle\cdot\langle x-0, y-0, z-0\rangle = 0$

$-x+y-z=0$

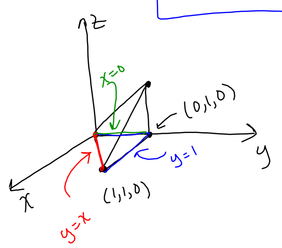
$z = y-x$

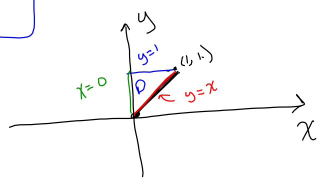
Type I triple integral

$dV = dz\, dA$

$0 \le z \le y-x$

D is in the xy plane





in xy plane, $z=0$, so $0=y-x$

$y=x$

define D:

Type 1: $0\le x\le 1$, $x\le y\le 1$ (dydx)

Type II: $0\le y\le 1$, $0\le x\le y$ (dx dy)

$\iiint_E xz\, dV \qquad dV = dz\, dx\, dy$

$$\int_0^1\int_0^y\left(\int_0^{y-x} xz\, dz\right)dx\, dy$$

$$\int_0^1\int_0^y x\frac{z^2}{2}\Big|_{z=0}^{z=y-x} dx\, dy$$

$$\frac{1}{2}\int_0^1\int_0^y x(y-x)^2\, dx\, dy$$

$$\frac{1}{2}\int_0^1\int_0^y x(y^2-2xy+x^2)\, dx\, dy$$

$$\frac{1}{2}\int_0^1\left(\int_0^y (xy^2-2x^2y+x^3)\, dx\right) dy$$

$$\frac{1}{2}\int_0^1\left[\frac{x^2y^2}{2}-\frac{2x^3}{3}y+\frac{x^4}{4}\right]_{x=0}^{x=y} dy$$

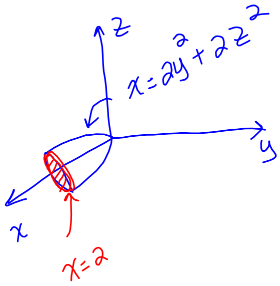
$$\frac{1}{2}\int_0^1 \frac{y^4}{2}-\frac{2y^4}{3}+\frac{y^4}{4}\, dy$$

$$\boxed{\frac{1}{120}}$$

Example 5: Evaluate $\iiint_E x\,dV$ where E is bounded by the paraboloid $x = 2y^2 + 2z^2$ and the plane $x = 2$.

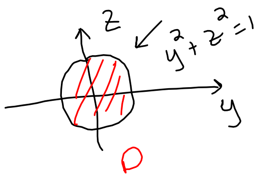
Define E:

$$2y^2+2z^2 \le x \le 2$$



$$\iiint_E x\,dV = \iint_D \left[\int_{2y^2+2z^2}^{2} x\,dx \right] dA$$

Define D: intersection of two surfaces



$$2y^2+2z^2 = 2$$
$$y^2+z^2=1$$

Define D in polar

$$0 \le r \le 1$$
$$0 \le \theta \le 2\pi$$

$\leftarrow dA = r\,dr\,d\theta$

Recall:

$$2y^2+2z^2 \le x \le 2$$
$$2(y^2+z^2) \le x \le 2$$
$$2r^2 \le x \le 2 \quad dx$$

$$\iint_D \left[\int_{2y^2+2z^2}^{2} x\,dx \right] dA$$

$$\int_0^{2\pi}\int_0^1\int_{2r^2}^{2} x\,dx\,r\,dr\,d\theta$$

$$\int_0^{2\pi}\int_0^1 \left[\frac{x^2}{2} \Big|_{x=2r^2}^{x=2} \right] r\,dr\,d\theta$$

$$\frac{1}{2}\int_0^{2\pi}\int_0^1 (4-4r^4)\,r\,dr\,d\theta$$

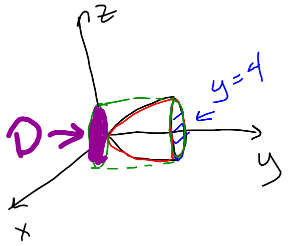
$$\left[\frac{1}{2}\int_0^{2\pi} d\theta\right]\int_0^1 (4r-4r^5)\,dr$$

$$\left(\frac{1}{2}\,\theta\Big|_0^{2\pi}\right)\left(2r^2-\frac{4r^6}{6}\right)\Big|_0^1$$

$$\frac{1}{2}(2\pi)\left(2-\frac{2}{3}\right) \qquad \boxed{\pi\left(\frac{4}{3}\right)}$$

Example 6: Evaluate $\iiint_E \sqrt{x^2+z^2}\,dV$ where E is the region bounded by the paraboloid $y = x^2 + z^2$ and the plane $y = 4$.

Define E:

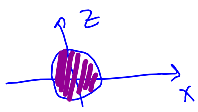


$x^2 + z^2 \le y \le 4$

$$\iiint_E \sqrt{x^2+z^2}\,dV \overset{*}{=} \iint_D \left[\int_{x^2+z^2}^{4} \sqrt{x^2+z^2}\,dy \right] dA$$

($\sqrt{x^2+z^2} = r$)

D is in the zx plane!



Find intersection of two surfaces

$x^2 + z^2 = 4$

define D in polar: $0 \le r \le 2$, $0 \le \theta \le 2\pi$ $\quad dA = r\,dr\,d\theta$

$x^2 + z^2 = r^2$

$$\int_0^{2\pi}\int_0^{2} \left[\int_{r^2}^{4} r\,dy \right] r\,dr\,d\theta$$

$x^2 + z^2 \le y \le 4$

$r^2 \le y \le 4$

$$\int_0^{2\pi}\int_0^{2} r y \Big|_{y=r^2}^{y=4} r\,dr\,d\theta$$

$$\int_0^{2\pi}\int_0^{2} r(4-r^2)\,r\,dr\,d\theta$$

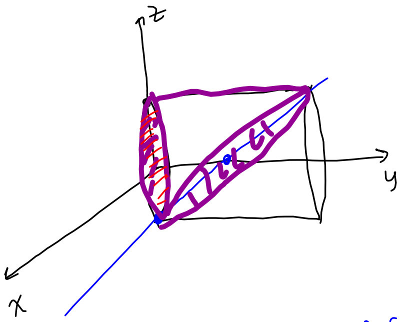
$$\int_0^{2\pi}\int_0^{2} (4r^2 - r^4)\,dr\,d\theta$$

$$\int_0^{2\pi} d\theta \int_0^{2} (4r^2 - r^4)\,dr$$

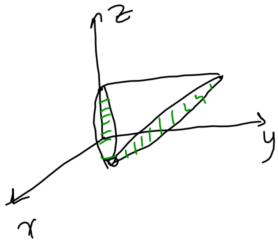
$$\theta\Big|_0^{2\pi} \left(\frac{4r^3}{3} - \frac{r^5}{5}\right)\Big|_0^{2}$$

$$(2\pi)\left(\frac{32}{3} - \frac{32}{5}\right)$$

Example 7: Set up but do not evaluate $\iiint_E x^2y^4z^4\,dV$ where E is the solid bounded by $4x^2+z^2=4$, $y=0$ and $y=z+2$.



$y = z+2$



$0 \le y \le z+2$

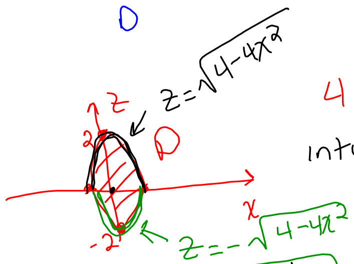
$$\iiint_E x^2y^4z^4\,dV = \iint_D \left[\int_0^{z+2} x^2y^4z^4\,dy\right] dA$$

D is in the zx plane

D is not polar! ellipse!

$4x^2+z^2=4$

$z = \pm\sqrt{4-4x^2}$



$4x^2+z^2=4$

into $y=0$

$y=z+2$

$dA = dz\,dx$

$-\sqrt{4-4x^2} \le z \le \sqrt{4-4x^2}$

$-1 \le x \le 1$

$$\iint_D \left[\int_0^{z+2} x^2y^4z^2\,dy\right] dA$$

$$\int_{-1}^{1}\int_{-\sqrt{4-4x^2}}^{\sqrt{4-4x^2}}\int_0^{z+2} x^2y^4z^2\,dy\,dz\,dx$$

Note: We can use a triple integral to find the volume of a solid E using the fact that if $f(x, y, z) = 1$ for all points in E, then the volume of E is $\iiint_E dV$.

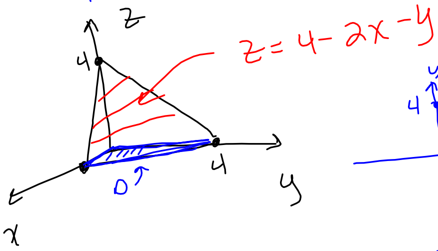
Example 8: Consider the tetrahedron enclosed by the three coordinate planes and the plane $2x + y + z = 4$. Set up but do not evaluate:

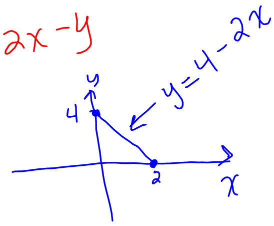
a.) a double integral that gives the volume of this solid.

Recall: If $f(x,y) \geq 0$ on D in the xy plane, then $\iint_D f(x,y)\,dA$ = Volume enclosed by $f(x,y)$, the x-axis on D.

$2x + y + z = 4$

$z = 4 - 2x - y$





$D: 0 \leq x \leq 2$
$\quad 0 \leq y \leq 4 - 2x$

$V = \iint_D f(x,y)\,dA$

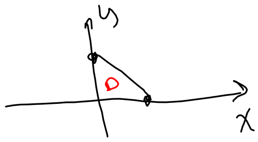
$= \int_0^2 \int_0^{4-2x} (4 - 2x - y)\,dy\,dx$

b.) a triple integral that gives the volume of this solid.

$V = \iiint_E dV$

$0 \leq z \leq 4 - 2x - y$

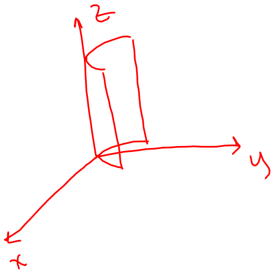
$\iiint_E dV = \iint_D \left[\int_0^{4-2x-y} dz \right] dA$



$D: 0 \leq x \leq 2$
$\quad 0 \leq y \leq 4 - 2x$

$= \int_0^2 \int_0^{4-2x} \int_0^{4-2x-y} dz\,dy\,dx$

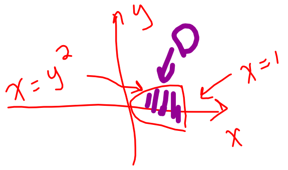
Example 9: Find the volume of the solid bounded by the cylinder $x = y^2$ and the planes $z = 0$ and $x + z = 1$.



Where is D?

in xy plane, $z=0$, $x=1$

$z = 1-x$
$z = 0$



$$V = \iiint_E dV$$

$$V = \iint_D \left[\int_{z=0}^{z=1-x} dz \right] dA$$

$D: \ -1 \le y \le 1$
$\quad y^2 \le x \le 1$

$$0 \le z \le 1-x$$

$$V = \int_{-1}^{1} \int_{y^2}^{1} \int_0^{1-x} dz\, dx\, dy$$

$$= \int_{-1}^{1} \int_{y^2}^{1} \left[z \Big|_{z=0}^{z=1-x} \right] dx\, dy$$

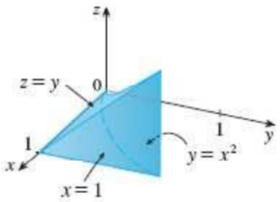
$$= \int_{-1}^{1} \int_{y^2}^{1} (1-x)\, dx\, dy$$

$$= \int_{-1}^{1} \left[\left(x - \frac{x^2}{2}\right) \Big|_{x=y^2}^{x=1} \right] dx\, dy$$

$$= \int_{-1}^{1} \left[1 - \frac{1}{2} - y^2 + \frac{y^4}{2} \right] dy$$

$$= \int_{-1}^{1} \left(\frac{1}{2} - y^2 + \frac{y^4}{2} \right) dy = \frac{8}{15}$$

Example 10: Rewrite $\int_0^1 \int_0^{x^2} \int_0^y f(x,y,z)\,dz\,dy\,dx$ as an iterated integral in a different order, integrating with respect to x, then z, then y by writing and graphing the projection of the solid E on the zy plane.

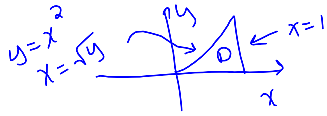


$0 \le z \le y$

$dz\,dy\,dx$

D: $0 \le y \le x^2$, $0 \le x \le 1$

their D is in xy plane



change to $dx\,dz\,dy$

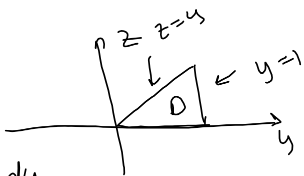
D in zy plane

$h_1(y,z) \le x \le h_2(y,z)$

$\sqrt{y} \le x \le 1$

$g_1(y) \le z \le g_2(y)$

$c \le y \le d$



D:

$z=0 \quad z=y$

$0 \le z \le y$

$0 \le y \le 1$

$$\int_0^1 \int_0^y \int_{\sqrt{y}}^1 f(x,y,z)\,dx\,dz\,dy$$