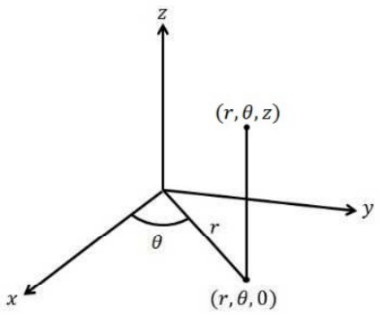
Section 15.7 Integration in Cylindrical Coordinates

Recall that in section 15.3 we introduced integration in the polar coordinate system. In three dimensions there are two coordinate systems that are similar to polar coordinates and give convenient descriptions of some commonly occuring surfaces and solids.

In the **Cylindrical Coordinate System**, a point P is represented by an ordered triple (r, θ, z) where r and θ are polar coordinates of the projection of the point P on onto the xy-plane and z is the directed distance from the xy-plane to P.

cylindrical point is (r, θ, z)



To convert from cylindrical to rectangular coordinates, we use the equations $x = r\cos\theta$, $y = r\sin\theta$, $z = z$.
To convert from rectangular to cylindrical coordinates, use the equations $x^2 + y^2 = r^2$, $\tan\theta = \frac{y}{x}$, $z = z$.

Example 1: Find the cylindrical coordinates of the rectangular coordinate $(\sqrt{3}, 1, 4)$. $= (x, y, z)$

convert to (r, θ, z)

$x^2 + y^2 = r^2$

$3 + 1 = r^2$

$\boxed{r = 2}$

$\tan\theta = \frac{y}{x}$

$\tan\theta = \frac{1}{\sqrt{3}}$

$\theta = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$

$\boxed{\theta = \frac{\pi}{6}}$

cylindrical point is $\left(2, \frac{\pi}{6}, 4\right)$

Example 2: Find the rectangular coordinates of the cylindrical coordinate $\left(2, \frac{2\pi}{3}, 5\right) = (r, \theta, z)$

convert to (x, y, z)

$x = r\cos\theta$

$x = 2\cos\frac{2\pi}{3}$

$x = 2\left(-\frac{1}{2}\right)$

$x = -1$

$y = r\sin\theta$

$y = 2\sin\frac{2\pi}{3}$

$y = 2\left(\frac{\sqrt{3}}{2}\right)$

$y = \sqrt{3}$

rectangular point is $(-1, \sqrt{3}, 5)$

Example 3: Write the equation in cylindrical coordinates

$\hookrightarrow r, \theta, z$

a.) $x^2 + y^2 + z^2 = 25$

$$\boxed{r^2 + z^2 = 25}$$

b.) $z = 12 - 4x^2 - 4y^2$

$z = 12 - 4(x^2 + y^2)$

$$\boxed{z = 12 - 4r^2}$$

c.) $z = \sqrt{3x^2 + 3y^2}$

$z = \sqrt{3(x^2 + y^2)}$

$$\boxed{z = \sqrt{3}\, r}$$

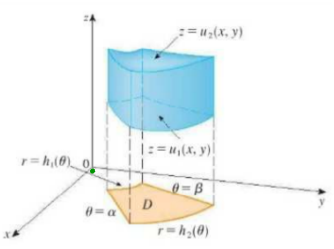
Triple Integrals in Cylindrical Coordinates

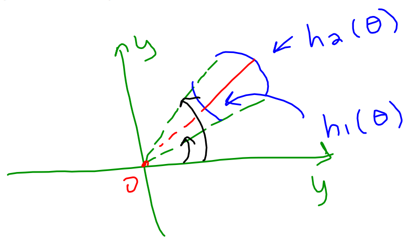
Suppose that E is a solid whose image D on the xy-plane can be described in **polar coordinates**:
$E = \{(x, y, z)|(x, y) \in D, \phi_1(x, y) \leq z \leq \phi_2(x, y)\}$ where D is given in polar coordinates by
$D = (r, \theta)|\alpha \leq \theta \leq \beta, h_1(\theta) \leq r \leq h_2(\theta)\}$.

We know from section 15.6 that $\iiint_E f(x, y, z)\, dV = \iint_D \left[\int_{\phi_1(x,y)}^{\phi_2(x,y)} f(x, y, z)\, dz \right] \boxed{dA.}$ Combining this with what we know from section 15.3, we get

$r\,dr\,d\theta$

$$\iiint_E f(x, y, z)\, dV = \int_{\theta=\alpha}^{\theta=\beta} \int_{r=h_1(\theta)}^{r=h_2(\theta)} \int_{z=\phi_1(r\cos\theta, r\sin\theta)}^{z=\phi_2(r\cos\theta, r\sin\theta)} f(r\cos\theta, r\sin\theta, z)\, dz\, r\, dr\, d\theta$$





Example 4: Evaluate $\int_{-3}^{3}\int_{0}^{\sqrt{9-x^2}}\int_{\sqrt{x^2+y^2}}^{3}(x^2+y^2)\,dz\,\boxed{dy\,dx}$ by converting to cylindrical coordinates.

$dA \to r\,dr\,d\theta$

what is D in polar?

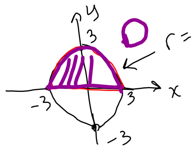
$\sqrt{x^2+y^2} \le z \le 3$

$$\boxed{\begin{array}{l} 0 \le y \le \sqrt{9-x^2} \\ -3 \le x \le 3 \end{array}}\ D$$

$y=0$

$y=\sqrt{9-x^2}$ $\quad y^2 = 9-x^2$

$x^2+y^2=9$



$0 \le y \le \sqrt{9-x^2}$

$-3 \le x \le 3$

D in polar:

$$\boxed{\begin{array}{l} 0 \le r \le 3 \\ 0 \le \theta \le \pi \\ r \le z \le 3 \end{array}}$$

integrand in polar: $x^2+y^2=r^2$

$$\boxed{\begin{array}{c}\text{integrand in polar:} \\ x^2+y^2=r^2\end{array}}$$

$$\int_0^{\pi}\int_0^{3}\int_r^{3} r^2\,dz\,r\,dr\,d\theta$$

$$\int_0^{\pi}\int_0^{3}\left[r^3 z\Big|_{z=r}^{z=3}\right]dr\,d\theta$$

$$\int_0^{\pi}\int_0^{3}(3r^3-r^4)\,dr\,d\theta$$

$$\int_0^{\pi}d\theta\int_0^{3}(3r^3-r^4)\,dr$$

$$\left(\theta\Big|_0^{\pi}\right)\left(\left(\frac{3r^4}{4}-\frac{r^5}{5}\right)\Big|_0^3\right)$$

$$(\pi)\left(\frac{3^5}{4}-\frac{3^5}{5}\right)$$

Example 5: Evaluate $\int_{0}^{2}\int_{0}^{\sqrt{4-y^2}}\int_{0}^{4-x^2-y^2} yz\,dz\,\underline{dx\,dy}$ by converting to cylindrical coordinates.

$0 \le z \le 4-x^2-y^2 \leftarrow$ (this in polar is $z=4-r^2$)

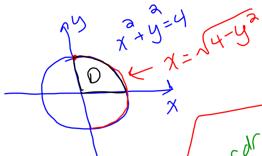
$$\boxed{\begin{array}{l} 0 \le x \le \sqrt{4-y^2} \\ 0 \le y \le 2 \end{array}}$$

D in xy plane should be polar.

$x=0$

$x=\sqrt{4-y^2}$ $\quad x^2=4-y^2$

$x^2+y^2=4$



D in polar is:

$$\boxed{\begin{array}{l} dA = r\,dr\,d\theta \quad 0 \le r \le 2 \\ 0 \le \theta \le \frac{\pi}{2} \\ 0 \le z \le 4-r^2 \\ \text{integrand is } yz \\ y = r\sin\theta \end{array}}$$

$$\int_0^{\frac{\pi}{2}}\int_0^{2}\int_0^{4-r^2}(r\sin\theta\, z)\,dz\,r\,dr\,d\theta$$

$$\int_0^{\frac{\pi}{2}}\int_0^{2}\int_0^{4-r^2}(r^2\sin\theta)\,z\,dz\,dr\,d\theta$$

$$\int_0^{\frac{\pi}{2}}\int_0^{2}\left[r^2\sin\theta\,\frac{z^2}{2}\Big|_{z=0}^{z=4-r^2}\right]dr\,d\theta$$

$$\frac{1}{2}\int_0^{\frac{\pi}{2}}\int_0^{2} r^2\sin\theta\,(4-r^2)^2\,dr\,d\theta$$

$r^2(4-r^2)^2$

$r^2(16-8r^2+r^4)$

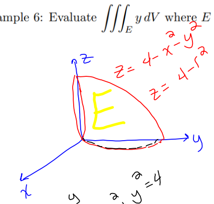
$16r^2-8r^4+r^6$

$$\frac{1}{2}\left(\int_0^{\frac{\pi}{2}}\sin\theta\,d\theta\right)\left(\int_0^{2} r^2(4-r^2)^2\right)d\theta$$

$$\frac{1}{2}\left[-\cos\theta\Big|_0^{\frac{\pi}{2}}\right]\int_0^{2}(16r^2-8r^4+r^6)\,dr$$

$$\frac{1}{2}\left[0-(-1)\right]\left[\frac{16}{3}r^3-\frac{8r^5}{5}+\frac{r^7}{7}\Big|_0^2\right] = \frac{512}{105}$$

Example 6: Evaluate $\iiint_E y\,dV$ where E is the bounded by $z = 4 - x^2 - y^2$ in the first octant.

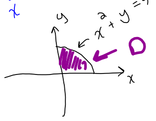


$dV = dz\,dA$ and D is polar in the xy plane → cylindrical integration

intersection of $z=4-x^2-y^2$ and the xy plane

$0 = 4 - x^2 - y^2$

$x^2+y^2=4$



D in polar: $0 \le r \le 2$, $0 \le \theta \le \frac{\pi}{2}$ $\quad dA = r\,dr\,d\theta$

$0 \le z \le 4 - r^2$

our integrand is y, in polar is $y = r\sin\theta$

$$\int_0^{\frac{\pi}{2}}\int_0^2\int_0^{4-r^2}(r\sin\theta)\,dz\,r\,dr\,d\theta$$

$$\int_0^{\frac{\pi}{2}}\int_0^2\left[r^2\sin\theta\cdot z\,\Big|_{z=0}^{z=4-r^2}\right]dr\,d\theta$$

$$\int_0^{\frac{\pi}{2}}\int_0^2 r^2\sin\theta\,(4-r^2)\,dr\,d\theta$$

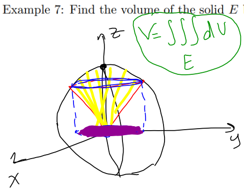
$$\left[\int_0^{\frac{\pi}{2}}\sin\theta\,d\theta\right]\left[\int_0^2(4r^2-r^4)\,dr\right]$$

$$\left(-\cos\theta\,\Big|_0^{\frac{\pi}{2}}\right)\left(\frac{4r^3}{3}-\frac{r^5}{5}\right)\Big|_0^2$$

$$\boxed{(1)\left(\frac{32}{3}-\frac{32}{5}\right)}$$

Example 7: Find the volume of the solid E bounded by $\underbrace{x^2+y^2+z^2=9}_{\text{sphere}}$ and $\underbrace{8z=x^2+y^2}_{\text{cone}}$.

$V = \iiint_E dV$



where do the two surfaces intersect?

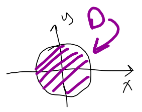
$\underbrace{x^2+y^2}_{8z}+z^2=9$

$8z+z^2=9$

$z^2+8z-9=0$

$(z+9)(z-1)=0$

$\cancel{z=-9}$ $\quad\boxed{z=1}$



$8 = x^2+y^2$

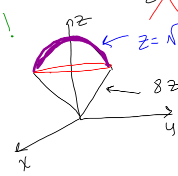
D is polar!

$0 \le r \le \sqrt{8}$

$0 \le \theta \le 2\pi$

$? \le z \le ?$

$\frac{r^2}{8} \le z \le \sqrt{9-r^2}$



$z=\sqrt{9-x^2-y^2}=\sqrt{9-r^2}$

$8z = x^2+y^2$

$z=\frac{r^2}{8}$

$$V = \iiint_E dV = \int_0^{2\pi}\int_0^{\sqrt{8}}\int_{\frac{r^2}{8}}^{\sqrt{9-r^2}} dz\,r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^{\sqrt{8}} z\,\Big|_{z=\frac{r^2}{8}}^{z=\sqrt{9-r^2}}\, r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^{\sqrt{8}}\left(\sqrt{9-r^2}-\frac{r^2}{8}\right)r\,dr\,d\theta$$

$$= \int_0^{2\pi}d\theta\int_0^{\sqrt{8}}\left(\underbrace{r\sqrt{9-r^2}}-\frac{r^3}{8}\right)dr$$

$u = 9-r^2$

$du = -2r\,dr$

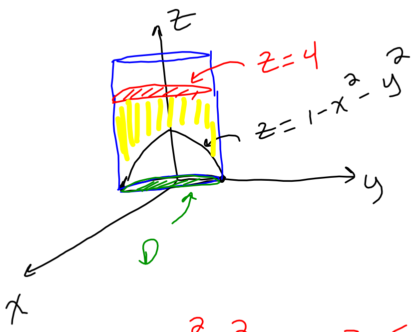
$-\frac{1}{2}\int\sqrt{u}\,du = -\frac{1}{2}\frac{2}{3}u^{3/2} = -\frac{1}{3}(9-r^2)^{3/2}$

$$= \theta\,\Big|_0^{2\pi}\left[-\frac{1}{3}(9-r^2)^{\frac{3}{2}}-\frac{r^4}{32}\right]\Big|_0^{\sqrt{8}}$$

$$=(2\pi)\left(-\frac{1}{3}(1)-\frac{(\sqrt{8})^4}{32}-\left(-\frac{1}{3}\cdot 9^{\frac{3}{2}}\right)\right)$$

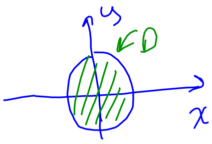
$$=\frac{40\pi}{3}$$

Example 8: Find $\iiint_E \sqrt{x^2+y^2}\,dV$ if E lies within the cylinder $x^2+y^2=1$, below the plane $z=4$ and above the paraboloid $z=1-x^2-y^2$.



$0 = 1-x^2-y^2$

$x^2+y^2=1$ circle!



describe D in polar

$0 \le r \le 1$

$0 \le \theta \le 2\pi$

$1-r^2 \le z \le 4$

$1-x^2-y^2 \le z \le 4$

$1-r^2 \le z \le 4$

integrand: $\sqrt{x^2+y^2} = r$

$$\iiint_E \sqrt{x^2+y^2}\,dV = \int_0^{2\pi}\int_0^1\int_{1-r^2}^4 r \, dz\, r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1\int_{1-r^2}^4 r^2\,dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 r^2 z \Big|_{z=1-r^2}^{z=4} dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 r^2\left(4-(1-r^2)\right)dr\,d\theta$$

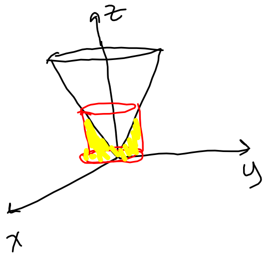
$$= \int_0^{2\pi}\int_0^1 r^2(3+r^2)\,dr\,d\theta$$

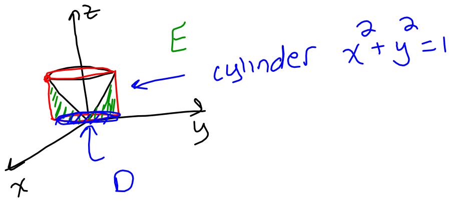
$$= \int_0^{2\pi} d\theta \int_0^1 (3r^2+r^4)\,dr$$

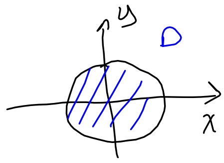
$$= \theta\Big|_0^{2\pi}\left(r^3+\frac{r^5}{5}\right)\Big|_0^1$$

$$= (2\pi)\left(1+\frac{1}{5}\right) = \frac{12\pi}{5}$$

Example 9: Evaluate $\iiint_E xy\,dV$ where E is the solid that lies with the cylinder $x^2+y^2=1$, above the plane $z=0$ below the cone $z=\sqrt{9x^2+9y^2}$. $= 3\sqrt{x^2+y^2}$







$0 \le r \le 1$

$0 \le \theta \le 2\pi$

$0 \le z \le 3r$

$0 \le z \le 3\sqrt{x^2+y^2}$

$0 \le z \le 3r$

integrand: $xy = r\cos\theta\, r\sin\theta$

$xy = r^2\cos\theta\sin\theta$

$dV = dz\, r\, dr\, d\theta$

$$\iiint_E xy\,dV = \int_0^{2\pi}\int_0^1\int_0^{3r} r^2\cos\theta\sin\theta\, dz\, r\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^1 (r^3\cos\theta\sin\theta)\, z\Big|_{z=0}^{z=3r} dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^1 r^3\cos\theta\sin\theta(3r)\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^1 3r^4\cos\theta\sin\theta\, dr\, d\theta$$

$$= \underbrace{\int_0^{2\pi}\cos\theta\sin\theta\, d\theta}\int_0^1 3r^4\, dr$$

$u = \sin\theta$ $\theta = 2\pi,\ u = 0$; $\theta = 0,\ u = 0$

$du = \cos\theta\, d\theta$

$$= \left(\int_0^0 u\,du\right)\left(\frac{3r^5}{5}\Big|_0^1\right)$$

$$= \boxed{0}$$