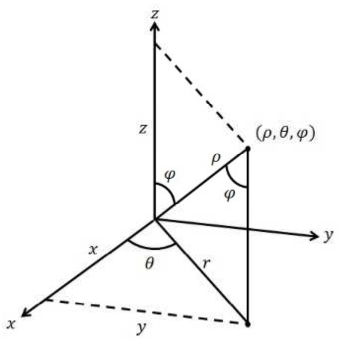
Section 15.8 Triple Integrals in Spherical Coordinates

In the **the Spherical Coordinate System**, a point P is represented by an ordered triple (ρ, θ, ϕ) where $\rho = |OP|$ is the distance from the origin to P, θ is the same angle as cylindrical coordinates, and ϕ is the angle between the positive z axis and the line segment OP. Note: $\rho \geq 0$ and $0 \leq \phi \leq \pi$.



The relationship between rectangular and spherical coordinates are $z = \rho\cos\phi$, $r = \rho\sin\phi$. But $x = r\cos\theta$, $y = r\sin\theta$, so $x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$ and $z = \rho\cos\phi$. Also, if we call $P(x, y, z)$, the distance formula gives us $\rho = \sqrt{x^2 + y^2 + z^2}$.

Example 1: Convert $(-1, \sqrt{3}, 2)$ from rectangular spherical coordinates.

Example 2: Convert $(4, \frac{\pi}{4}, \frac{\pi}{6})$ from spherical to rectangular coordinates.

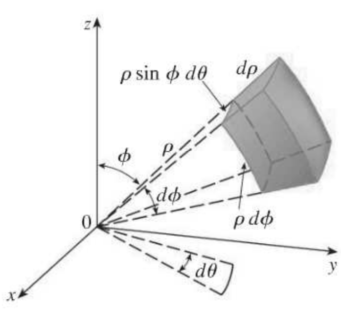
Example 3: Write the equation in spherical coordinates.

a.) $x^2 + y^2 + z^2 = 25$

b.) $z = 12 - 4x^2 - 4y^2$

Triple Integrals in Spherical Coordinates

Recall we defined the spherical coordinates (ρ, θ, ϕ) where $\rho = |OP|$ is the distance from the origin to P, θ is the same angle as cylindrical coordinates, and ϕ is the angle between the positive z axis and the line segment OP. Note: $\rho \geq 0$ and $0 \leq \phi \leq \pi$. Also, the relationship between rectangular and spherical coordinates are $z = \rho \cos \phi$, $r = \rho \sin \phi$. But $x = r \cos \theta$, $y = r \sin \theta$, so $x = \rho \sin \phi \cos \theta$, $y = \rho \sin \phi \sin \theta$ and $z = \rho \cos \phi$. Also, if we call $P(x, y, z)$, the distance formula gives us $\rho = \sqrt{x^2 + y^2 + z^2}$.



In this coordinate system, the equivalent of a box is is a spherical wedge. To integrate over such a system, we have

$$\iiint_E f(x, y, z)\, dV = \int_{\theta=c}^{\theta=d} \int_{\phi=\alpha}^{\phi=\beta} \int_{\rho=a}^{\rho=b} f(\rho \sin \phi \cos \theta, \rho \sin \phi \sin \theta, \rho \cos \phi)\, \rho^2 \sin \phi d\rho d\phi d\theta$$

Example 4: Evaluate $\int_{-3}^{3}\int_{0}^{\sqrt{9-x^2}}\int_{0}^{\sqrt{9-x^2-y^2}} z\sqrt{x^2+y^2+z^2}\,dz\,dy\,dx$ by converting to spherical coordinates.

Example 5: Evaluate $\iiint_E \sqrt{x^2+y^2+z^2}\,dV$ where E is the region between the spheres $x^2+y^2+z^2=1$ and $x^2+y^2+z^2=9$.

Example 6: Evaluate $\iiint_E xz\, dV$ where E is bounded above by the sphere $x^2 + y^2 + z^2 = 4$ and below by the cone $z = \sqrt{x^2 + y^2}$.

Example 7: Use spherical coordinates to find the volume of the part of the sphere $\rho \leq 4$ that lies between the cones $\phi = \dfrac{\pi}{6}$ and $\phi = \dfrac{\pi}{3}$.

Example 8: Find the volume of the solid that lies within the sphere $x^2 + y^2 + z^2 = 4$, above the xy plane and below the cone $z = \sqrt{x^2 + y^2}$.