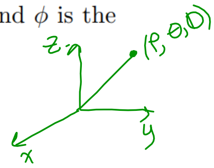
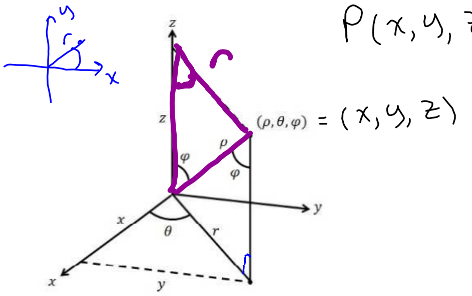
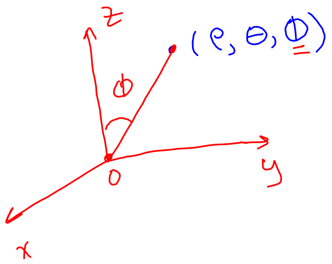
Section 15.8 Triple Integrals in Spherical Coordinates

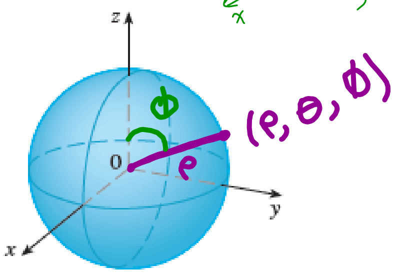
In the **the Spherical Coordinate System**, a point P is represented by an ordered triple (ρ, θ, ϕ) where $\rho = |OP|$ is the distance from the origin to P, θ is the same angle as cylindrical coordinates, and ϕ is the angle between the positive z axis and the line segment OP. Note: $\rho \geq 0$ and $\boxed{0 \leq \phi \leq \pi}$.



$P(x,y,z) = (\rho, \theta, \phi)$







$\cos\phi = \dfrac{z}{\rho}$

$\boxed{z = \rho\cos\phi}$

$\sin\phi = \dfrac{r}{\rho} \rightarrow r = \rho\sin\phi$

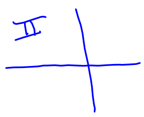
From polar, $\boxed{x = r\cos\theta = \rho\sin\phi\cos\theta}$

$\boxed{y = r\sin\theta = \rho\sin\phi\sin\theta}$

$(x,y,z) = (\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)$

The relationship between rectangular and spherical coordinates are $z = \rho\cos\phi$, $r = \rho\sin\phi$. But $x = r\cos\theta$, $y = r\sin\theta$, so $x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$ and $z = \rho\cos\phi$. Also, if we call $P(x, y, z)$, the distance formula gives us $\rho = \sqrt{x^2 + y^2 + z^2}$.

Example 1: Convert $(-1, \sqrt{3}, 2)$ from rectangular spherical coordinates.



$x = -1, \ y = \sqrt{3}, \ z = 2$ Find (ρ, θ, ϕ)

$\rho^2 = x^2 + y^2 + z^2$

$\rho^2 = 3 + 1 + 4$

$\boxed{\rho = \sqrt{8} = 2\sqrt{2}}$

$\tan\theta = \dfrac{y}{x} = \dfrac{\sqrt{3}}{-1}$ $\theta = \arctan(-\sqrt{3})$

$\boxed{\theta = \dfrac{2\pi}{3}}$

$z = \rho\cos\phi$

$2 = 2\sqrt{2}\cos\phi$

$\cos\phi = \dfrac{1}{\sqrt{2}}$ $\phi = \dfrac{\pi}{4}$

$\begin{matrix} \rho & \theta & \phi \end{matrix}$

$\left(2\sqrt{2}, \dfrac{2\pi}{3}, \dfrac{\pi}{4}\right)$

$\rho \quad \theta \quad \phi$

Example 2: Convert $(4, \frac{\pi}{4}, \frac{\pi}{6})$ from spherical to rectangular coordinates.

$\rho = 4$

$\theta = \frac{\pi}{4}$

$\phi = \frac{\pi}{6}$

$x = \rho \sin\phi \cos\theta = 4 \cdot \frac{1}{2} \cdot \frac{\sqrt{2}}{2}$

$y = \rho \sin\phi \sin\theta = 4 \cdot \frac{1}{2} \frac{\sqrt{2}}{2}$

$z = \rho \cos\phi = 4 \frac{\sqrt{3}}{2}$

$\boxed{(\sqrt{2}, \sqrt{2}, 2\sqrt{3})}$

~~Example 3: Convert $(1, \frac{\pi}{2}, 1)$ from cylindrical to spherical coordinates~~.

omit

~~Example 4: Convert $\left(4, \frac{\pi}{4}, \frac{\pi}{3}\right)$ from spherical to cylindrical coordinates.~~ omit

Example ~~1~~ 3: Write the equation in spherical coordinates.

a.) $x^2 + y^2 + z^2 = 25$

$\rho^2 = 25$

$\rho = 5$

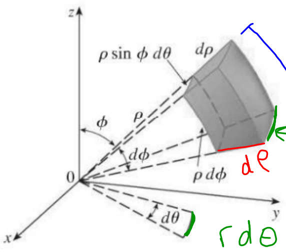
b.) $z = 12 - 4x^2 - 4y^2$ $r = \rho \sin \phi$

$z = 12 - 4r^2$

$\boxed{z = 12 - 4\rho^2 \sin^2 \phi}$

Triple Integrals in Spherical Coordinates

Recall we defined the spherical coordinates (ρ, θ, ϕ) where $\rho = |OP|$ is the distance from the origin to P, θ is the same angle as cylindrical coordinates, and ϕ is the angle between the positive z axis and the line segment OP. Note: $\rho \geq 0$ and $0 \leq \phi \leq \pi$. Also, the relationship between rectangular and spherical coordinates are $z = \rho \cos\phi$, $r = \rho \sin\phi$. But $x = r\cos\theta$, $y = r\sin\theta$, so $x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$ and $z = \rho\cos\phi$. Also, if we call $P(x, y, z)$, the distance formula gives us $\rho = \sqrt{x^2 + y^2 + z^2}$.



$\rho\, d\phi$

$$\iiint_E f(x,y,z)\, dV$$

dV = volume of a wedge

$d\rho$

$\rho \sin\phi\, d\theta$

$r\, d\theta$

$\rho \sin\phi\, d\theta$

$L = r\theta$

$dV = \rho \sin\phi\, \rho\, d\rho\, d\phi\, d\theta$

$dV = \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$

In this coordinate system, the equivalent of a box is is a spherical wedge. To integrate over such a system, we have

$$\iiint_E f(x,y,z)\, dV = \int_{\theta=c}^{\theta=d} \int_{\phi=\alpha}^{\phi=\beta} \int_{\rho=a}^{\rho=b} f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\, \rho^2 \sin\phi d\rho d\phi d\theta$$

Example 4: Evaluate $\int_{-3}^{3}\int_0^{\sqrt{9-x^2}}\int_0^{\sqrt{9-x^2-y^2}} z\sqrt{x^2+y^2+z^2}\,\underline{dz\,dy\,dx}$ by converting to spherical coordinates.

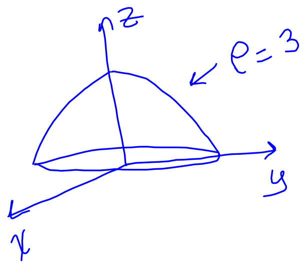
$$\boxed{0 \le z \le \sqrt{9-x^2-y^2}}$$

$$\boxed{\begin{array}{l} 0 \le y \le \sqrt{9-x^2} \\ -3 \le x \le 3 \end{array}}$$

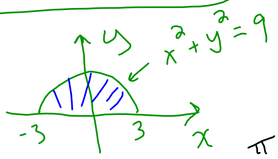
$z=0,\quad z=\sqrt{9-x^2-y^2}$

$z^2 = 9-x^2-y^2$

$x^2+y^2+z^2=9$



$$\boxed{\begin{array}{l} 0 \le \rho \le 3 \\ 0 \le \phi \le \frac{\pi}{2} \end{array}}$$



$$\boxed{0 \le \theta \le \pi}$$

$$\int_0^{\pi}\int_0^{\frac{\pi}{2}}\int_0^{3} \underbrace{(\rho\cos\phi)(\rho)}_{\text{integrand}}\ \underbrace{\rho^2\sin\phi\, d\rho\, d\phi\, d\theta}_{dV}$$

Fubini!

$$\left(\int_0^{\pi} d\theta\right)\left(\int_0^{\frac{\pi}{2}} \underline{\cos\phi\sin\phi\, d\phi}\right)\left(\int_0^3 \rho^4\right) d\rho$$

$u=\sin\phi$

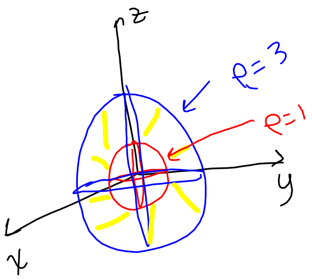
$du=\cos\phi\, d\phi$

$\int u\,du = \frac{u^2}{2}$

$$\left(\theta\Big|_0^{\pi}\right)\left(\frac{1}{2}\sin^2\phi\Big|_0^{\frac{\pi}{2}}\right)\left(\frac{\rho^5}{5}\Big|_0^{3}\right)$$

$$(\pi)\left(\frac{1}{2}\right)\left(\frac{243}{5}\right) = \boxed{\frac{243\pi}{10}}$$

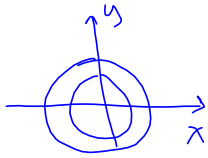
Example 5. Evaluate $\iiint_E \sqrt{x^2+y^2+z^2}\,dV$ where E is the region between the spheres $x^2+y^2+z^2=1$ and $x^2+y^2+z^2=9$.



$1 \le \rho \le 3$

$0 \le \phi \le \pi$

$0 \le \theta \le 2\pi$



$$\int_0^{2\pi}\int_0^{\pi}\int_1^{3} (\rho)(\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta)$$

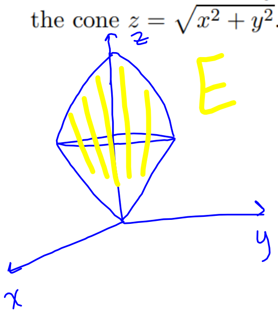
$$\int_0^{2\pi} d\theta \int_0^{\pi} \sin\phi \, d\phi \int_1^{3} \rho^3 \, d\rho$$

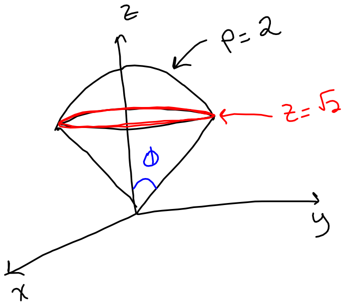
$$\left(\theta \Big|_0^{2\pi}\right)\left(-\cos\phi \Big|_0^{\pi}\right) \frac{\rho^4}{4}\Big|_1^3$$

$$(\cancel{2}\pi)(\cancel{2})\left(\cancel{\tfrac{1}{4}}(81-1)\right)$$

$$\boxed{80\pi}$$

Example ~~5~~ 6 Evaluate $\iiint_E xz\,dV$ where E is bounded above by the sphere $x^2+y^2+z^2=4$ and below by the cone $z=\sqrt{x^2+y^2}$.





$$0 \le \rho \le 2$$
$$0 \le \phi \le ? = \frac{\pi}{4}$$
$$0 \le \theta \le 2\pi$$

$$\underbrace{x^2+y^2}_{z^2}+z^2=4, \quad z=\sqrt{x^2+y^2} \to z^2=x^2+y^2$$

$$2z^2=4 \to z=\sqrt{2}$$

recall: $z=\rho\cos\phi$

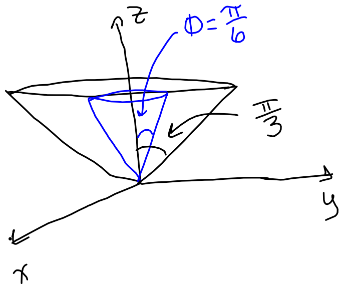
$$\sqrt{2}=2\cos\phi$$
$$\phi=\frac{\pi}{4}$$

$$\iiint_E xz\,dV = \int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_0^{2}(\rho\sin\phi\cos\theta)(\rho\cos\phi)\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

$$=\underbrace{\int_0^{2\pi}\cos\theta\,d\theta}\ \underbrace{\int_0^{\frac{\pi}{4}}\sin^2\phi\cos\phi\,d\phi}_{\left.\frac{\sin^3\phi}{3}\right|_0^{\frac{\pi}{4}}}\ \underbrace{\int_0^2\rho^4\,d\rho}_{\left.\frac{\rho^5}{5}\right|_0^2} = 0$$

$$\sin\theta\Big|_0^{2\pi}=0$$

Example ~~9~~ 7: Use spherical coordinates to find the volume of the part of the sphere $\rho \leq 4$ that lies between the cones $\phi = \dfrac{\pi}{6}$ and $\phi = \dfrac{\pi}{3}$.

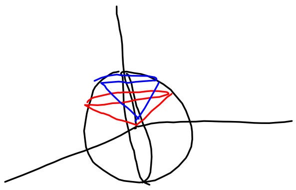


recall: Volume of $E = \iiint_E dV$

$$\frac{\pi}{6} \leq \phi \leq \frac{\pi}{3}$$

$$0 \leq \rho \leq 4$$

$$0 \leq \theta \leq 2\pi$$



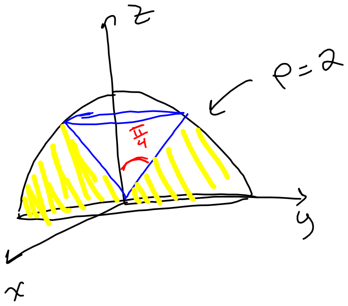
$$V = \iiint_E dV = \int_0^{2\pi}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\int_0^4 \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \int_0^{2\pi} d\theta \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sin\phi \, d\phi \int_0^4 \rho^2 \, d\rho$$

$$= \theta \Big|_0^{2\pi} \left[-\cos\phi \Big|_{\frac{\pi}{6}}^{\frac{\pi}{3}} \right] \frac{\rho^3}{3} \Big|_0^4$$

$$V = (2\pi)\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}\right)\left(\frac{64}{3}\right)$$

Example ~~10~~ 8: Find the volume of the solid that lies within the sphere $x^2 + y^2 + z^2 = 4$, above the xy plane and below the cone $z = \sqrt{x^2 + y^2}$.



$0 \le \rho \le 2$

$\frac{\pi}{4} \le \phi \le \frac{\pi}{2}$

$0 \le \theta \le 2\pi$

$$V(E) = \iiint dV = \int_0^{2\pi} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^{2} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \int_0^{2\pi} d\theta \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin\phi \, d\phi \int_0^2 \rho^2 \, d\rho$$

$$= \theta \Big|_0^{2\pi} \left[-\cos\phi \Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}} \right] \frac{\rho^3}{3} \Big|_0^2$$

$$\boxed{(2\pi)\left(\frac{\sqrt{2}}{2}\right)\left(\frac{8}{3}\right)}$$