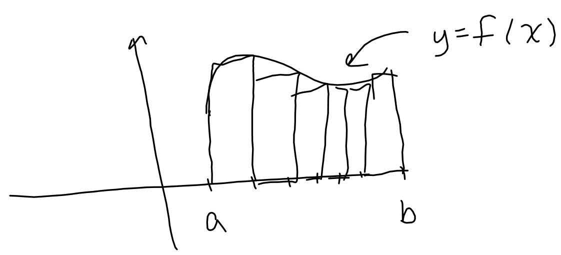
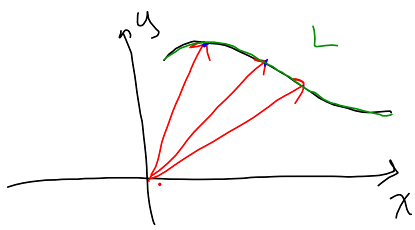
$y=f(x)$, $\int_a^b f(x)\,dx$



Recall: if $x=x(t)$
$y=y(t)$



$\boxed{r(t)=\langle x(t), y(t)\rangle}$ $\rightarrow$ $r'(t)=\left\langle \frac{dx}{dt}, \frac{dy}{dt}\right\rangle$

$$L=\int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$$

$$|r'(t)|=\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}$$

$$\boxed{L=\int_a^b |r'(t)|\,dt}$$

Section 16.2 Line Integrals

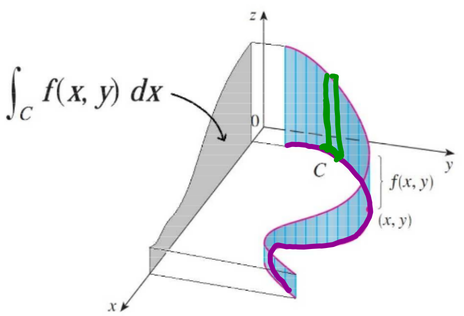
Recall from Calculus 2 that the arc length of a curve defined parametrically by

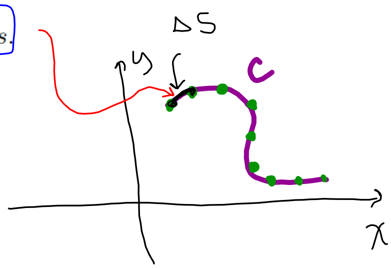
$x = x(t)$, $y = y(t)$, $a \le t \le b$ is $L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_a^b |\mathbf{r}'(t)|\, dt$.

In this section, we will define an integral that is similar to a single integral except that instead of integrating a function $f(x)$ along an interval $[a, b]$, we will integrate a surface $f(x, y)$ over a curve C in the xy-plane defined parametrically by $\mathbf{r(t)} = \langle x(t), y(t)\rangle$. These integrals are called **line integrals.**

Instead of partitioning along the x-axis, we are partitioning the curve C defined parametrically by $x = x(t)$, $y = y(t)$, $a \le b \le b$. Thus the length of each 'sub arc' is approximitely $\Delta s = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$.

Thus the area of a typical polygon is is $\boxed{f(x, y) * \Delta s}$.





If we define the norm of the partition $||P||$ to be the length of the smallest subarc, then The **line integral of f along C** is $\int_C f(x, y)ds = \lim_{||P|| \to 0} \sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta s_i$, if the limit exists.

Definition: If f is defined on a smooth curve C defined as $\mathbf{r}(t) = \langle x(t), y(t)\rangle$, then the **line integral of f along C** is

$$\int_C f(x, y)ds = \int_a^b (f(x(t), y(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_a^b (f(x(t), y(t))|\mathbf{r}'(t)|\, dt$$

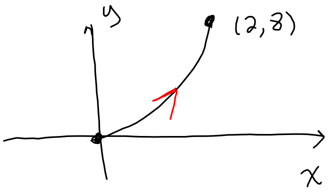
When $f(x, y) \ge 0$, the line integral of f along C represents the area of one side of the "fence" or "curtain" whose base is C and whose height at any point on the curve is $f(x, y)$.

Note: **For these integrals, the orientation of the curve, which direction is is traversed, is important.** If C and $-C$ represent traversing the same curve but in different directions, then

$$\int_{-C} f(x, y)dx = -\int_C f(x, y)dx.$$

Before we begin, since the parametrization is not always given explicitly, let's practice parameterizing curves.

(i) Parameterize the curve $y = x^3$, from the point $(0, 0)$ to the point $(2, 8)$ in two different ways.



$y = f(x)$ $\quad x = t$ or $y = t$

$x = \sqrt[3]{y}$

$y = x^3$

① $r_1(t) = \langle t, t^3 \rangle, \quad 0 \le t \le 2$

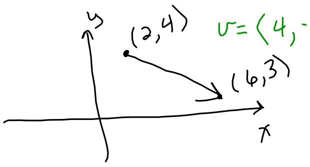
if $y = t$

$x = \sqrt[3]{t}$

② $r_2(t) = \langle \sqrt[3]{t}, t \rangle, \quad 0 \le t \le 8$

(ii) Parameterize the line passing going from the point $(2, 4)$ towards the point $(6, 3)$.

To parameterize a line use $r(t) = \vec{r}_0 + t\vec{v}$



$r(t) = \langle 2, 4 \rangle + t \langle 4, -1 \rangle$

$\boxed{r(t) = \langle 2 + 4t, 4 - t \rangle \quad 0 \le t \le 1}$

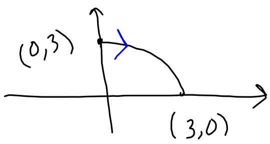
t will always be $0 \le t \le 1$

why does this work? $t = 0$ $\quad r(0) = \langle 2, 4 \rangle$

$t = 1$ $\quad r(1) = \langle 6, 3 \rangle$

(iii) Parameterize the circle $x^2 + y^2 = 9$ in the first quadrant, orientated clockwise.

To parametrize a circle



put sine into the x position
cosine into the y position

$x = 3\sin\theta$

$y = 3\cos\theta$

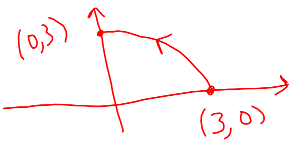
$r(t) = \langle 3\sin\theta, 3\cos\theta \rangle, \quad 0 \le \theta \le \frac{\pi}{2}$

$\theta = 0$: $\quad r(0) = \langle 0, 3 \rangle$ ✓

$\theta = \frac{\pi}{2}$: $\quad r(\frac{\pi}{2}) = \langle 3, 0 \rangle$ ✓

if same problem, but counterclockwise:

cosine in x position
sine in y position



$r(t) = \langle 3\cos\theta, 3\sin\theta \rangle, \quad 0 \le \theta \le \frac{\pi}{2}$

check: $r(0) = \langle 3, 0 \rangle$ ✓

$r(\frac{\pi}{2}) = \langle 0, 3 \rangle$ ✓

integrating with respect to arc length

Example 1: $\int_C y\,ds$, where C is defined as $\mathbf{r}(t) = \langle 2t, t^3\rangle$, $0 \le t \le 1$.

Recall: If C is parameterized by $\boxed{r(t)}$, $a \le t \le b$,

then $\int_C f(x,y)\,ds = \int_a^b f(r(t))\underbrace{|r'(t)|\,dt}_{ds}$

$f(x,y) = y$, $\boxed{r(t) = \langle 2t, t^3\rangle}$, $r'(t) = \langle 2, 3t^2\rangle$

$0 \le t \le 1$ $\qquad |r'(t)| = \sqrt{4+9t^4}$

$$\int_C y\,ds = \int_a^b f(r(t))|r'(t)|\,dt$$

$$= \int_0^1 [t^3]\sqrt{4+9t^4}\,dt$$

$u = 4+9t^4$

$du = 36t^3\,dt$

$$\frac{1}{36}\int_4^{13} \sqrt{u}\,du$$

$$\frac{1}{36}\,\frac{2}{3}\,u^{\frac{3}{2}}\Big|_4^{13}$$

$$\boxed{\frac{1}{54}\left(13^{\frac{3}{2}} - 4^{\frac{3}{2}}\right)}$$

Example 2: $\int_C (x^2 + y)\, ds$ where C consists of the line segment from the point $(2, 8)$ to $(0, 15)$.

$r(t) = r_0 + tv$

$= \langle 2, 8 \rangle + t\langle -2, 7 \rangle$

$r(t) = \boxed{\langle 2-2t, 8+7t \rangle}$ $\quad 0 \le t \le 1$

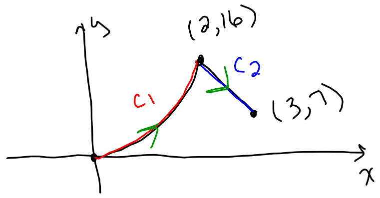
$r'(t) = \langle -2, 7 \rangle$, so $|r'(t)| = \sqrt{53}$

$\int_C (x^2+y)\, ds = \int_0^1 \underbrace{\left((2-2t)^2 + 8+7t\right)}_{f(r(t))} \underbrace{\sqrt{53}\, dt}_{|r'(t)|dt}$

$\sqrt{53} \int_0^1 (12 - t + 4t^2)\, dt = \boxed{\sqrt{53}\ \frac{77}{6}}$

Definition: If C is a **piecewise-smooth curve**, that is C is made up a collection of smooth curves where one curve ends then the next curve begins, then the line integral of f along C is defined to be the sum of the integrals of f along each smooth piece of C.

Example 3: Set up but do not evaluate $\int_C (2+x^2y)ds$, where C is the arc of the curve $y=x^4$ from $(0,0)$ to $(2,16)$ and then the line segment from the point $(2,16)$ to the point $(3,7)$.



Parameterize C_1: $x=t, \quad y=t^4$

C_1 $\begin{cases} r_1(t)=\langle t, t^4\rangle, \quad 0\le t\le 16 \\ r_1'(t)=\langle 1, 4t^3\rangle, \quad |r_1'(t)|=\sqrt{1+16t^6} \end{cases}$

Parameterize C_2: Line from $(2,16)$ to $(3,7)$

$r_2(t)=r_0+tv$
$\quad =\langle 2,16\rangle + t\langle 1,-9\rangle, \quad 0\le t\le 1$

$\left.\begin{array}{l} r_2(t)=\langle 2+t, 16-9t\rangle \end{array}\right\} C_2$

$r_2'(t)=\langle 1,-9\rangle$ so $|r_2'(t)|=\sqrt{82}$

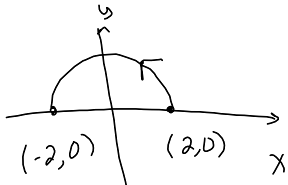
$$\int_C (2+x^2y)\,ds$$

$$=\int_{C_1}(2+x^2y)\,ds+\int_{C_2}(2+x^2y)\,ds$$

$$=\underbrace{\int_0^{16}(2+t^2t^4)\sqrt{1+16t^6}\,dt}_{\text{along } C_1}+\underbrace{\int_0^1\left(2+(2+t)^2(16-9t)\right)\sqrt{82}\,dt}_{\text{along } C_2}$$

Example 4: $\int_C (x^2 y)ds$, where C is the top half of the circle $x^2 + y^2 = 4$, oriented counterclockwise.

$$\int_C x^2 y\, ds = \int_{?}^{?} f(r(\theta))\,|r'(\theta)|\, d\theta$$



counterclockwise: $\boxed{r(\theta) = \langle 2\cos\theta,\ 2\sin\theta \rangle,}$ $0 \le \theta \le \pi$

$r'(\theta) = \langle -2\sin\theta,\ 2\cos\theta \rangle$

$|r'(\theta)| = \sqrt{4\sin^2\theta + 4\cos^2\theta}$

$= 2$

Example 4: $\int_C (x^2 y)ds$, where C is the top half of the circle $x^2 + y^2 = 4$, oriented counterclockwise.

$$\int_C x^2 y\, ds = \int_0^{\pi} \underbrace{(4\cos^2\theta)(2\sin\theta)}_{f(r(\theta))} \cdot \underbrace{2}_{|r'(\theta)|} d\theta$$

$$16\int_0^{\pi} \cos^2\theta \sin\theta\, d\theta$$

$u = \cos\theta$

$du = -\sin\theta\, d\theta$

$$-16\int_1^{-1} u^2\, du = 16\int_{-1}^{1} u^2\, du$$

$$= 16\, \frac{u^3}{3}\Big|_{-1}^{1}$$

$$= \frac{16}{3}\left(1 - (-1)\right) = \boxed{\frac{32}{3}}$$

Definition: Let C be a smooth curve defined by the parametric equations $x = x(t)$, $y = y(t)$ for $a \le t \le b$. The **line integral of f along** C **with respect to** x is $\int_C f(x,y)dx = \int_a^b (f(x(t), y(t))\ x'(t)\ dt.$ The **line integral of f along** C **with respect to** y is $\int_C f(x,y)dy = \int_a^b (f(x(t), y(t))\ y'(t)\ dt$

Example 5: Evaluate $\int_C y^2 dx + x dy$, where C is decribed by $\mathbf{r}(t) = \langle e^t, 2e^{2t} \rangle$, $0 \le t \le 2$.

$$x = e^t, \quad y = 2e^{2t}$$

$$dx = e^t\,dt, \quad dy = 4e^{2t}\,dt$$

$$\int_C y^2 dx + x\,dy = \int_0^2 \left[\underbrace{\left(4e^{4t}\right)}_{y^2} \underbrace{\left(e^t\,dt\right)}_{dx} + e^t \cdot 4e^{2t}\,dt \right]$$

$$= \int_0^2 \left(4e^{5t} + 4e^{3t}\right) dt$$

$$= \left(\frac{4}{5}e^{5t} + \frac{4}{3}e^{3t}\right)\Big|_0^2$$

$$= \boxed{\frac{4}{5}e^{10} + \frac{4}{3}e^{6} - \left(\frac{4}{5} + \frac{4}{3}\right)}$$

~~**Definition:** If C is a **piecewise-smooth curve**, that is C is made up a collection of smooth curves where one curve ends then the next curve begins, then the line integral of f along C is defined to be the sum of the integrals of f along each smooth piece of C.~~

Example 6: Evaluate $\int_C y^2 dx + x dy$, where C is the arc of the parabola $x = 4 - y^2$ from $(-5, -3)$ to $(3, 1)$.

Parameterize C: $y = t,\ x = 4 - t^2$

$r(t) = \langle t, 4 - t^2 \rangle, \quad -3 \le t \le 1$

$y = t, \quad x = 4 - t^2$

$dy = dt, \quad dx = -2t\,dt$

$$\int_C y^2 dx + x\,dy = \int_{-3}^{1} \left[t^2(-2t\,dt) + (4 - t^2)(dt) \right]$$

$$= \int_{-3}^{1} (-2t^3 + 4 - t^2)\,dt$$

$$= \boxed{\frac{140}{3}}$$

Line Integrals in Space:

Let $f(x,y,z)$ be a function defined on a smooth curve C defined by $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$ for $a \le t \le b$. The line integral of f along C is

$$* \int_C f(x,y,z)ds = \int_a^b (f(x(t),y(t),z(t))\ |\mathbf{r}'(t)|\,dt = \int_a^b (f(x(t),y(t),z(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt$$

Example 7: $\int_C xy^2z^2ds$ where C is the line segment going from $(1,1,0)$ to $(2,3,1)$.

Parameterize C : $r(t) = \vec{r}_0 + t\vec{v}$

$= \langle 1,1,0\rangle + t\langle 1,2,1\rangle, \quad 0 \le t \le 1$

$r(t) = \langle 1+t, 1+2t, t\rangle, \quad r'(t) = \langle 1,2,1\rangle$

$|r'(t)| = \sqrt{6}$

$$\int_C xy^2z^2ds = \int_0^1 (1+t)(1+2t)^2(t^2)\sqrt{6}\,dt$$

$$= \sqrt{6}\int_0^1 (t^2+t^3)(1+4t+4t^2)\,dt$$

$$= \sqrt{6}\int_0^1 (t^2+5t^3+8t^4+4t^5)\,dt$$

$$= \boxed{\sqrt{6}\,\frac{77}{20}}$$

Definition: Let C be a smooth curve defined by the parametric equations $x = x(t)$, $y = y(t)$, $z = z(t)$ for $a \leq t \leq b$. The **line integral of f along** C **with respect to** x is $\int_C f(x,y,z)dx = \int_a^b (f(x(t),y(t),z(t))\ x'(t)\ dt$.

The **line integral of f along** C **with respect to** y is $\int_C f(x,y,z)dy = \int_a^b (f(x(t),y(t),z(t))\ y'(t)\ dt$. The **line integral of f along** C **with respect to** z is $\int_C f(x,y,z)dz = \int_a^b (f(x(t),y(t),z(t))\ z'(t)\ dt$.

Example 8: Evaluate $\int_C ydz + zdy + xdx$ where C: $x = t^4$, $y = t^3$, $z = t^4$, $0 \leq t \leq 1$.

$dx = 4t^3 dt,\quad dy = 3t^2 dt,\quad dz = 4t^3 dt$

$$\int_0^1 (t^3)(4t^3 dt) + (t^4)\,3t^2 dt + (t^4)(4t^3)dt$$

$$\int_0^1 \left(4t^6 + 3t^6 + 4t^7\right) dt = \boxed{\frac{3}{2}}$$

Line Integrals over vector fields: Suppose now are moving a particle along a curve C through a vector (force) field, $\mathbf{F}$. We define the **line integral of F along C** to be $\boxed{\int_c \mathbf{F}\cdot d\mathbf{r}} = \int_a^b (\mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)dt$

Example 9: Find $\int_c \mathbf{F}\cdot d\mathbf{r}$, C: $\mathbf{r}(\mathbf{t}) = \langle t, t^2, t^4\rangle$, $0 \le t \le 1$, and $\mathbf{F}(x,y,z) = \langle x, z^2, -4y\rangle$.

$x=t,\ y=t^2,\ z=t^4$

$r'(t) = \langle 1, 2t, 4t^3\rangle$

plug $r(t) = \langle t, t^2, t^4\rangle$ into $F(x,y,z) = \langle x, z^2, -4y\rangle$

$$\int_c F\cdot dr = \int_0^1 \underbrace{\langle t, t^8, -4t^2\rangle}_{F(r(t))}\cdot\underbrace{\langle 1, 2t, 4t^3\rangle}_{r'(t)}\,dt$$

$$= \int_0^1 (t + 2t^9, -16t^5)\,dt = \boxed{-\frac{59}{30}}$$

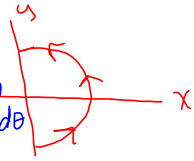
Think of this as moving a particle through a bunch of 'forces' with 'magnitudes', so we must work against the forces in order to move the particle along the curve C. Thus the line integral of $\mathbf{F}$ along C represents the work done in moving the object along a curve through a force field.

Definition: Let $\mathbf{F}$ be a continuous vector (force) field defined on a smooth curve C given by a vector function $\mathbf{r(t)}$, $a \leq t \leq b$. The work W done in moving an object along a curve C given by $\mathbf{r(t)} = \langle x(t), y(t), z(t)\rangle$ is

$$W = \int_c \mathbf{F} \cdot d\mathbf{r} = \int_a^b (\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)dt$$

Example 10: Find the work done by the force field $\mathbf{F}(x, y) = \langle x^2, xy\rangle$ in moving an object counterclockwise around the right half of the circle $x^2 + y^2 = 4$.

Parameterize C:



$$W = \int_C F \cdot dr = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \langle 4\cos^2\theta, 4\cos\theta\sin\theta\rangle \cdot \langle -2\sin\theta, 2\cos\theta\rangle d\theta$$

$r(t) = \langle 2\cos\theta, 2\sin\theta\rangle$,
$-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$

$r'(t) = \langle -2\sin\theta, 2\cos\theta\rangle$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \langle 4\cos^2\theta, 4\cos\theta\sin\theta\rangle \cdot \langle -2\sin\theta, 2\cos\theta\rangle d\theta$$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (-8\cos^2\theta\sin\theta + 8\cos^2\theta\sin\theta)d\theta = \boxed{0}$$

Example 11: Find the work done by the force field $\mathbf{F}(x, y, z) = \langle x + y, x + z, y + z\rangle$ in moving an object from the point $(2, 2, 4)$ to the point $(4, 5, 8)$.

$$r(t) = \vec{r}_0 + t\vec{v}$$

$$= \langle 2,2,4\rangle + t\langle 2,3,4\rangle, \quad 0 \le t \le 1$$

$$\vec{r}(t) = \langle 2+2t,\ 2+3t,\ 4+4t\rangle$$

$$r'(t) = \langle 2, 3, 4\rangle$$

$$W = \int_C F \cdot dr = \int_0^1 \langle 2+2t+2+3t,\ 2+2t+4+4t,\ 2+3t+4+4t\rangle \cdot \langle 2,3,4\rangle\, dt$$

$$= \int_0^1 \langle 4+5t,\ 6+6t,\ 6+7t\rangle \cdot \langle 2,3,4\rangle\, dt$$

$$= \int_0^1 (8+10t+18+18t+24+28t)\, dt$$

$$\int_0^1 (56t+50)\, dt = \boxed{66}$$