Section 16.2 Line Integrals

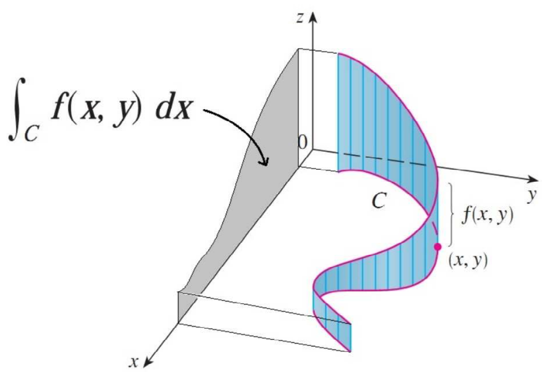
Recall from Calculus 2 that the arc length of a curve defined parametrically by

$x = x(t)$, $y = y(t)$, $a \le t \le b$ is $L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = \int_a^b |\mathbf{r}'(t)|\,dt.$

In this section, we will define an integral that is similar to a single integral except that instead of integrating a function $f(x)$ along an interval $[a, b]$, we will integrate a surface $f(x, y)$ over a curve C in the xy-plane defined parametrically by $\mathbf{r(t)} = \langle x(t), y(t)\rangle$. These integrals are called **line integrals.**

Instead of partitioning along the x-axis, we are partitioning the curve C defined parametrically by $x = x(t)$, $y = y(t)$, $a \le b \le b$. Thus the length of each 'sub arc' is approximitely $\Delta s = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$.

Thus the area of a typical polygon is is $f(x, y) * \Delta s$.



If we define the norm of the partition $||P||$ to be the length of the smallest subarc, then The **line integral of f along C** is $\int_C f(x, y)ds = \lim_{||P|| \to 0} \sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta s_i$, if the limit exists.

Definition: If f is defined on a smooth curve C defined as $\mathbf{r}(t) = \langle x(t), y(t)\rangle$, then the **line integral of f along C** is

$$\int_C f(x, y)ds = \int_a^b (f(x(t), y(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = \int_a^b (f(x(t), y(t))|\mathbf{r}'(t)|\,dt$$

When $f(x, y) \ge 0$, the line integral of f along C represents the area of one side of the "fence" or "curtain" whose base is C and whose height at any point on the curve is $f(x, y)$.

Note: **For these integrals, the orientation of the curve, which direction is is traversed, is important.** If C and $-C$ represent traversing the same curve but in different directions, then

$\int_{-C} f(x, y)dx = -\int_C f(x, y)dx.$

Before we begin, since the parametrization is not always given explicitly, let's practice parameterizing curves.

(i) Parameterize the curve $y = x^3$, from the point $(0, 0)$ to the point $(2, 8)$ in two different ways.

(ii) Parameterize the line passing going from the point $(2, 4)$ towards the point $(6, 3)$.

(iii) Parameterize the circle $x^2 + y^2 = 9$ in the first quadrant, orientated clockwise.

Example 1: $\int_C y ds$, where C is defined as $\mathbf{r}(t) = \langle 2t, t^3 \rangle$, $0 \leq t \leq 1$.

Example 2: $\int_C (x^2 + y)\, ds$ where C consists of the line segment from the point $(2, 8)$ to $(0, 15)$.

Definition: If C is a **piecewise-smooth curve**, that is C is made up a collection of smooth curves where one curve ends then the next curve begins, then the line integral of f along C is defined to be the sum of the integrals of f along each smooth piece of C.

Example 3: Set up but do not evaluate $\int_C (2 + x^2 y)ds$, where C is the arc of the curve $y = x^4$ from $(0, 0)$ to $(2, 16)$ and then the line segment from the point $(2, 16)$ to the point $(3, 7)$.

Example 4: $\int_C (x^2 y) ds$, where C is the top half of the circle $x^2 + y^2 = 4$, oriented counterclockwise.

Definition: Let C be a smooth curve defined by the parametric equations $x = x(t)$, $y = y(t)$ for $a \leq t \leq b$. The **line integral of f along** C **with respect to** x is $\int_C f(x,y)dx = \int_a^b (f(x(t), y(t))\ x'(t)\ dt$. The **line integral of f along** C **with respect to** y is $\int_C f(x,y)dy = \int_a^b (f(x(t), y(t))\ y'(t)\ dt$

Example 5: Evaluate $\int_C y^2 dx + x dy$, where C is decribed by $\mathbf{r}(t) = \langle e^t, 2e^{2t} \rangle$, $0 \leq t \leq 2$.

Definition: If C is a **piecewise-smooth curve**, that is C is made up a collection of smooth curves where one curve ends then the next curve begins, then the line integral of f along C is defined to be the sum of the integrals of f along each smooth piece of C.

Example 6: Evaluate $\int_C y^2 dx + x dy$, where C is the arc of the parabola $x = 4 - y^2$ from $(-5, -3)$ to $(3, 1)$.

Line Integrals in Space:

Let $f(x, y, z)$ be a function defined on a smooth curve C defined by $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$ for $a \leq t \leq b$. The line integral of f along C is

$$\int_C f(x,y,z)ds = \int_a^b (f(x(t), y(t), z(t))\ |\mathbf{r}'(t)|\, dt = \int_a^b (f(x(t), y(t), z(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\ dt$$

Example 7: $\int_C xy^2z^2 ds$ where C is the line segment going from $(1, 1, 0)$ to $(2, 3, 1)$.

Definition: Let C be a smooth curve defined by the parametric equations $x = x(t)$, $y = y(t)$, $z = z(t)$ for $a \leq t \leq b$. The **line integral of f along C with respect to x** is $\int_C f(x, y, z)dx = \int_a^b (f(x(t), y(t), z(t))\ x'(t)\ dt$.

The **line integral of f along C with respect to y** is $\int_C f(x, y, z)dy = \int_a^b (f(x(t), y(t), z(t))\ y'(t)\ dt$. The **line integral of f along C with respect to z** is $\int_C f(x, y, z)dz = \int_a^b (f(x(t), y(t), z(t))\ z'(t)\ dt$.

Example 8: Evaluate $\int_C ydz + zdy + xdx$ where C: $x = t^4$, $y = t^3$, $z = t^4$, $0 \leq t \leq 1$.

Line Integrals over vector fields: Suppose now are moving a particle along a curve C through a vector (force) field, $\mathbf{F}$. We define the **line integral of F along C** to be $\int_c \mathbf{F}\cdot d\mathbf{r} = \int_a^b (\mathbf{F}(\mathbf{r}(t))\cdot \mathbf{r}'(t)dt$

Example 9: Find $\int_c \mathbf{F}\cdot d\mathbf{r}$, C: $\mathbf{r}(\mathbf{t}) = \langle t, t^2, t^4\rangle$, $0 \le t \le 1$, and $\mathbf{F}(x,y,z) = \langle x, z^2, -4y\rangle$.

Think of this as moving a particle through a bunch of 'forces' with 'magnitudes', so we must work against the forces in order to move the particle along the curve C. Thus the line integral of $\mathbf{F}$ along C represents the work done in moving the object along a curve through a force field.

Definition: Let $\mathbf{F}$ be a continuous vector (force) field defined on a smooth curve C given by a vector function $\mathbf{r(t)}$, $a \leq t \leq b$. The work W done in moving an object along a curve C given by $\mathbf{r(t)} = \langle x(t), y(t), z(t) \rangle$ is

$$W = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b (\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt$$

Example 10: Find the work done by the force field $\mathbf{F}(x, y) = \langle x^2, xy \rangle$ in moving an object counterclockwise around the right half of the circle $x^2 + y^2 = 4$.

Example 11: Find the work done by the force field $\mathbf{F}(x, y, z) = \langle x + y, x + z, y + z \rangle$ in moving an object from the point $(2, 2, 4)$ to the point $(4, 5, 8)$.