Section 16.3 Fundamental Theorem for Line Integrals

In section 16.2, we learned how to find a line integral over a vector field **F** along a curve C that is parametrized by $\mathbf{r}(t)$, $a \le t \le b$: $\int_c \mathbf{F}\cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t))\cdot \mathbf{r}'(t)\,dt$.

Example 1: Suppose we are moving a particle from the point $(0,0)$ to the point $(2,4)$ in a force field $\mathbf{F}(x,y) = \langle y^2, x\rangle$. Find $\int_c \mathbf{F}\cdot d\mathbf{r}$ where:

a.) The particle travels along the line segment from $(0,0)$ to $(2,4)$.

b.) The particle travels along the curve $y = x^2$ from $(0,0)$ to $(2,4)$.

Note: Although the end points are the same, the value of the line integral is **different** because the **paths** are different. In this section, we will learn under what conditions the line integral is independent of the path taken.

Definition: If **F** is a continuous vector field, we say that $\int_C \mathbf{F}\cdot d\mathbf{r}$ is **independent of path** if and only if $\int_{C_1} \mathbf{F}\cdot d\mathbf{r} = \int_{C_2} \mathbf{F}\cdot d\mathbf{r}$ for any two paths C_1 and C_2 with the same starting and ending points. In other words, the line integral is the same **no matter what path** you travel on as long as the endpoints are the same.

Definition: A vector field **F** is called a **conservative vector field** if it is the gradient of some scalar funtion f, that is there exists a function f so that $\mathbf{F} = \nabla f$. We call f the **potential function**.

Example 2. Consider $f(x,y) = x^2y - y^3$. Find the gradient and explain why it is conservative. What is the potential function?

Recall the Fundamental Theorem of Calculus tells us that $\int_a^b f'(x)dx = f(b) - f(a)$. Since $\nabla f = \langle f_x, f_y \rangle$, we can think of the potential function, f, as some sort of antiderivative of ∇f. Hence $\int \mathbf{F} \cdot d\mathbf{r} = \int \nabla f \cdot d\mathbf{r}$.

Fundamental Theorem for Line Integrals: Let C be a smooth curve given by the vector function $\mathbf{r}(t)$, $a \leq t \leq b$. Let $\mathbf{F}$ be a conservative vector field. Let f be a differentiable function of two or three variables whose gradient vector, ∇f, is continuous on C. Then

$$\int_c \mathbf{F} \cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$$

Note: **Line integrals of conservative vectors fields are independent of path** because in a conservative vector field, the line integral is computed by only using the **endpoints** of the domain! Therefore, if we are in a conservative vector field, the line integral along a curve C in that vector field will be the same **no matter what curve we travel across** that connects the endpoints together. **WHICH MEANS WE DON'T EVEN NEED TO PARAMETERIZE THE CURVE!**

Note: If $\mathbf{F}$ is a **conservative** vector field and C is a **closed path** (one in which the starting point and ending point is the same), then $\int_c \mathbf{F} \cdot d\mathbf{r} = 0$. Why?

Example 3: Let $f(x, y) = 3x + x^2y - yx^2$. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F} = \nabla f$ and C is the curve given by $\mathbf{r}(t) = \langle 2t, t^2 \rangle$, $1 \leq t \leq 2$.

Question: How do we determine if a vector field is conservative, and if so, how do we find the potential function? The 'test for conservative' we use depends on whether **F** is in $\mathbb{R}^2$ or $\mathbb{R}^3$.

Theorem: $\mathbf{F}(x,y) = \langle P(x,y), Q(x,y)\rangle = P\mathbf{i} + Q\mathbf{j}$ is a conservative vector field, where P and Q have continuous first-order partial derivatives on a domain D, if and only if $\dfrac{\partial Q}{\partial x} = \dfrac{\partial P}{\partial y}$. Note: This above criteria to determine if a vector field is conservative works only for $\mathbb{R}^2$.

Short proof:

Example 4: Is $\mathbf{F}(x,y) = \langle 3x^2 - 4y, 4y^2 - 2x\rangle$ a conservative vector field? If so, find a function f so that $\mathbf{F} = \nabla f$.

Example 5: Is $\mathbf{F}(x,y) = \langle x + y, x - 2\rangle$ a conservative vector field? If so, find a function f so that $\mathbf{F} = \nabla f$.

Example 6: Given $\mathbf{F}(x, y) = \langle 2xy^3, 3x^2y^2 \rangle$. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ where C is the curve given by

$\mathbf{r}(t) = \langle t^3 + 2t^2 - t, 3t^4 - t^2 \rangle$, $0 \le t \le 2$.

Example 7: Let $\mathbf{F}(x, y) = \langle 3 + 2xy^2, 2x^2y \rangle$. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ where C is the arc of the hyperbola $y = \dfrac{1}{x}$ from $(1, 1)$ to $\left(4, \dfrac{1}{4}\right)$.

Example 8: Given $\mathbf{F}(x, y) = \langle 3x^2 - 4y, 4y^2 - 2x \rangle$. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ where C is the curve given by

$\mathbf{r}(t) = \langle t^3 + 1, t^2 \rangle$, $0 \leq t \leq 1$.

Example 9: Given that $\mathbf{F} = \langle 4xe^z, \cos(y), 2x^2e^z \rangle$ is conservative and $\mathbf{r}(t) = \langle \sin(t), t, \cos(t) \rangle$, compute $\int_C \mathbf{F} \cdot d\mathbf{r}$ for $0 \leq t \leq \frac{\pi}{2}$. Note: We had to tell you $\mathbf{F}$ is conservative since we have not yet learned the testing criteria for conservativness in $\mathbb{R}^3$